Online Appendix for A Market Based Solution for Fire Sales and Other Pecuniary Externalities (Not For Publication)

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June 12, 2018

This appendix contains omitted formulations, proofs, and results for the paper titled "A Market Based Solution for Fire Sales and Other Pecuniary Externalities" by Weerachart T. Kilenthong and Robert M. Townsend.

A Proofs of the First Welfare Theorem and the Existence Theorem for the General Economy in the Main Text

For convenience we write again the mixture representation for competitive equilibrium and the planner problem. To overcome a potential non-convexity problem, we use the mixture representation for the proof, as in Prescott and Townsend (1984a). Let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be the fraction of agents type h assigned to a bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$, where the component at state s of the right to trade in an exchange $\mathbf{p}$ is defined by

$$\Delta_s^h(\mathbf{p}) \equiv \tau_{1s}^{h*}\left(p_s, k^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \mathbf{e}_s^h\right), \forall s, \tag{A.1}$$

which is the standard excess demand for good 1 in the spot market s at $t = 1$ for an agent type h holding collateral $k^h(\mathbf{p})$, securities $\boldsymbol{\theta}^h(\mathbf{p})$, and being in an exchange $\mathbf{p}$.

This x^h puts mass on the entire bundle including $\mathbf{p}$. This is a way we economize the notation relative to the discrete choice $\delta^h(\mathbf{p})$ of the text. This notation also allows a positive mass between zero and one, with more than one active exchange, as shown in a numerical example in online Appendix F.5.3. From the individual point of view, a non-degenerate x^h is a lottery but in the aggregate there is no uncertainty, which we utilize in the broker-dealer problem below.

At the individual level, for each agent type h, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0$ denote the probability of receiving period $t = 0$ consumption $\mathbf{c}_0$, collateral k, securities $\boldsymbol{\theta}$, period $t = 1$ spot trades $\boldsymbol{\tau}$, and being in exchanges indexed by $\mathbf{p} \equiv [p_s]_s$ with rights to trade $\boldsymbol{\Delta}$. For convenience, we write here again the spot market budget and the collateral constraints in state s:

$$\tau_{1s} + p_s \tau_{2s} = 0, \forall s, \tag{A.2}$$

$$p_s R_s k^h + \sum_j D_{js} \theta_j^h \geq 0, \forall s. \tag{A.3}$$

Accordingly, we impose the following condition on a probability measure:

$$x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \geq 0 \text{ if } (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \text{ satisfies } (A.1), (A.2), (A.3), \tag{A.4}$$

and zero otherwise. The consumption possibility set of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}_+^n : \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) = 1, \text{ and } (A.4) \text{ holds} \right\}. \tag{A.5}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option.

With all choice objects gridded up as an approximation, the commodity space L is assumed to be a finite n-dimensional linear space. We prove the existence and welfare theorems for a given, gridded space. But this is not at all essential. The limiting arguments under the weak-topology used in Prescott and Townsend (1984a) can be applied to establish existence and the welfare theorems for lotteries or measures on continuum spaces; even though the commodity space L is not finite, it is a separable metric space (Parthasarathy, 1967).

For notational purposes, let $\mathbf{w} \equiv (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ be a typical bundle, and the utility derived from it for an agent type h is defined by $U^h(\mathbf{w}) = u^h\big(c_{10}, c_{20}\big) + \sum_s \pi_s u^h\big(e^h_{1s} + \sum_j D_{js}\theta^h_j + \tau_{1s}, e^h_{2s} + R_s k + \tau_{2s}\big)$. Then, we have the maximization problem for agents as part of the definition of equilibrium: for each h, $\mathbf{x}^h \in X^h$ solves

$$\max_{\mathbf{x}^h} \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w}) \tag{A.6}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 0$ budget constraint, that the valuation of endowments sold provides revenue for purchase of the lotteries.

$$\sum_{\mathbf{w}} P(\mathbf{w})\, x^h(\mathbf{w}) \quad \leq \quad e^h_{10} + p_0 e^h_{20}, \tag{A.7}$$

taking price of good 2 at $t = 0$, p_0, and prices of lottery, $P(\mathbf{w})$ as given.

We introduce broker dealers that run the $\mathbf{p}$-platforms and deal with households for trades in securities, collateral, rights to trade and spot trades. The consumption $\mathbf{c}_0$ and collateral k commitments are sold but must be funded by the requisite amount of consumption goods and collateral. Securities, rights and spot trades do not require resources but are cleared by the broker-dealers. There are constant returns to scale in these activities so it is as if there were one representative broker-dealer. See Prescott and Townsend (1984a) for the introduction of broker-dealer.

Formally, the broker-dealer issues (sells) $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \in \mathbb{R}_+$ units of each bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$, at the unit price $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. Note that $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ at a particular bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ is the number of units of that bundle. There is nothing random. $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ simply measures the quantity of the particular bundle sold. Another distinct bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}', \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ has its own quantity, number of units $b(\mathbf{c}_0, k, \boldsymbol{\theta}', \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. With $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, the intermediary is taking distinct positions in the market. The clearing constraints below will ensure that when we add up over all bundles, the net positions add up to zero.

Let $\mathbf{b} \in L$ be the vector of the number of bundles issued as one move across the underlying commodity points $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$. With constant returns to scale (see below), the profit of a broker-dealer must be zero and the number of broker-dealers becomes indeterminate. Therefore, without loss of generality, we act as if there were one representative broker-dealer, which takes prices as given.

The broker-dealer takes prices $p_0, P(\mathbf{w})$ as given and supplies $\mathbf{b}$ to solve the following profit maximization problem:

$$\max_{\mathbf{b}} \sum_{\mathbf{w}} b(\mathbf{w}) \left[P(\mathbf{w}) - c_{10} - p_0 c_{20} - p_0 k\right] \tag{A.8}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \theta_j = 0, \ \forall j; \mathbf{p}, \tag{A.9}$$

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \tau_{\ell s} = 0, \ \forall s; \ell = 1, 2; \mathbf{p}, \tag{A.10}$$

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) \Delta_s = 0 \ , \forall s; \mathbf{p}, \tag{A.11}$$

where $\widehat{Q}_j(\mathbf{p})$, $\hat{p}_\ell(\mathbf{p}, s)$ and $\widehat{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (A.9)-(A.11), respectively.

The existence of a maximum to the broker-dealer's problem requires, that for any bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$,

$$P(\mathbf{w}) \leq c_{10} + p_0 \left[c_{20} + k\right] + \sum_j \widehat{Q}_j(\mathbf{p}) \theta_j + \sum_s \sum_\ell \hat{p}_\ell(\mathbf{p}, s) \tau_{\ell s} + \sum_s \widehat{P}_\Delta(\mathbf{p}, s) \Delta_s. \tag{A.12}$$

where $\widehat{Q}_j(\mathbf{p})$, $\hat{p}_\ell(\mathbf{p}, s)$ and $\widehat{P}_\Delta(\mathbf{p}, s)$ are the Lagrange multipliers for constraints (A.9)-(A.11), respectively. This is where the prices of active and inactive exchanges come from. Condition (A.12) holds with equality if $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) > 0$. In this case the revenue per unit trade in the $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})$ is equal to the sum of these shadow costs times the quantity of commitments in the particular objects. On the other hand, if the inequality (A.12) is strict, then $b(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta}) = 0$ as when implicit shadow costs are greater than revenue. We can still price the inactive exchanges by setting (A.12) at equality.

Market clearing conditions in the two consumption goods is standard, purchased consumptions and collateral by the broker-dealer equals supply of endowments from the households:

$$\sum_{\mathbf{w}} b(\mathbf{w})\, c_{10} = \sum_{h} \alpha^h e^h_{10}, \tag{A.13}$$

$$\sum_{\mathbf{w}} b(\mathbf{w})\, [c_{20} + k] = \sum_{h} \alpha^h e^h_{20}. \tag{A.14}$$

The net demand for contracts by households, allowing non-degenerate fractions in the population, equals the supply of contracts by the broker-dealer:

$$\sum_{h} \alpha^h x^h(\mathbf{w}) = b(\mathbf{w}),\ \forall \mathbf{w}. \tag{A.15}$$

See online Appendix F.5.3 for a particular clarified example of what broker-dealers in the context of an environment with multiple active exchanges.

Definition A.1. *A competitive equilibrium with rights to trade (with mixtures) is a specification of allocation* $(\mathbf{x}^h, \mathbf{b})$*, and prices* $(p_0, P(\mathbf{w}))$ *such that*

(i) for each h, $\mathbf{x}^h \in X^h$ *solves the utility maximization problem (A.6) taking prices as given;*

(ii) for the broker-dealer, $\mathbf{b}$ *solves the profit maximization problem (A.8), taking prices as given;*

(iii) market clearing conditions (A.13)-(A.15) hold.

The Planner Problem in the Mixture/Lottery Representation

The Pareto problem with Pareto weights $\left[\lambda^h\right]_h$ is defined as follows.

$$\max_{\left[\mathbf{x}^h\right]_h} \sum_{h} \lambda^h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w}) \tag{A.16}$$

subject to

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, c_{10} = \sum_h \alpha^h e^h_{10}, \tag{A.17}$$

$$\sum_h \alpha^h \sum_{\mathbf{w}} x^h(\mathbf{w})\, [c_{10} + k] = \sum_h \alpha^h e^h_{20}, \tag{A.18}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \tau_{\ell s} = 0, \forall \ell; s; \mathbf{p}, \tag{A.19}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \theta_j = 0, \forall j; \mathbf{p}, \tag{A.20}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{p}, \boldsymbol{\Delta})\, \Delta_s = 0, \forall s; \mathbf{p}. \tag{A.21}$$

Proof of the First Welfare Theorem 1. This proof follows Prescott and Townsend (1984a). Let allocations $(\mathbf{x}, \mathbf{b})$, and prices $P(\mathbf{w})$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e., there is an attainable allocation $\tilde{\mathbf{x}}$ such that $\sum_{\mathbf{w}} \tilde{x}^h(\mathbf{w})\, U^h(\mathbf{w}) \geq \sum_{\mathbf{w}} x^h(\mathbf{w})\, U^h(\mathbf{w})$ for all h and $\sum_{\mathbf{w}} \tilde{x}^{\hat{h}}(\mathbf{w})\, U^{\hat{h}}(\mathbf{w}) > \sum_{\mathbf{w}} x^{\hat{h}}(\mathbf{w})\, U^{\hat{h}}(\mathbf{w})$ for some $\hat{h}$. With local non-satiation of preferences, $\sum_{\mathbf{w}} P(\mathbf{w})\, x^h(\mathbf{w}) \leq \sum_{\mathbf{w}} P(\mathbf{w})\, \tilde{x}^h(\mathbf{w})$ for all h, and $\sum_{\mathbf{w}} P(\mathbf{w})\, x^{\hat{h}}(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w})\, \tilde{x}^{\hat{h}}(\mathbf{w})$ for some $\hat{h}$. Summing over all agents with weights α^h, we have

$$\sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h x^h(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w}) \sum_h \alpha^h \tilde{x}^h(\mathbf{w}). \tag{A.22}$$

In addition, for each allocation $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we can find a corresponding supply from the intermediary such that $b(\mathbf{w}) = \sum_h \alpha^h x^h(\mathbf{w})$ and $\tilde{b}(\mathbf{w}) = \sum_h \alpha^h \tilde{x}^h(\mathbf{w})$. Since both $\mathbf{x}$ and $\tilde{\mathbf{x}}$ satisfy all feasibility conditions, $\mathbf{b}$ and $\tilde{\mathbf{b}}$ both satisfy the clearing constraints (A.9)-(A.11). As a result, (A.22) can be rewritten as$\sum_{\mathbf{w}} P(\mathbf{w})\, b(\mathbf{w}) < \sum_{\mathbf{w}} P(\mathbf{w})\, \tilde{b}(\mathbf{w})$. On the other hand, the intermediary's profit maximization implies that $\sum_{\mathbf{w}} P(\mathbf{w})\, b(\mathbf{w}) \geq \sum_{\mathbf{w}} P(\mathbf{w})\, \tilde{b}(\mathbf{w})$. This is a contradiction! □

Proof of the Existence Theorem 3. For notational convenience, we put the endowment $\mathbf{e}^h$ onto the grid. Let $\mathbf{P} = [P(\mathbf{w})]_{\mathbf{w}}$ be the prices of all bundles. As in Prescott and Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{ \mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P} \cdot \mathbf{P}} \leq 1 \right\}, \tag{A.23}$$

where "$\cdot$" is the inner product operator. Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$. The local non-satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^* \cdot \mathbf{P}^*}.$$

Note that $\mathbf{P}' \cdot \mathbf{P}' = 1$. In order to preserve the convexity of the mapping with prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max\left\{0, \lambda^h + \frac{\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)}{A}\right\}, \tag{A.24}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h}, \tag{A.25}$$

where A is a positive number such that $\sum_h \left|\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)\right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$ is a mapping from $S^{H-1} \times \overline{X} \times S^{n-1} \longrightarrow S^{H-1} \times \overline{X} \times S^{n+1}$. Since each set is non-empty, compact, and convex, so is its cross-product. In addition, the overall mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda, \mathbf{x}, \mathbf{P})$.

Proved in Theorem 2, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the nonsatiation and the positive endowment assumptions ensure that there is a cheaper point as in the proof of Theorem 2. As a result, a compensated equilibrium is a competitive equilibrium with transfers.

We now need to show that there is no need for wealth transfers in equilibrium, i.e., the budget constraint without transfers $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ holds for every agent h. It is not difficult to show that $\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint (without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. □

B Definition of Competitive Equilibrium (with Externality) Corresponding to the General Model in the Main Text

Definition A.2 (Competitive Equilibrium with Externality). *A competitive equilibrium is a specification of prices* $(p_0, \mathbf{Q}, \mathbf{p})$*, and an allocation* $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ *such that*

- *for any agent type h as a price taker,* $\left(\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ *solves*

$$\max_{\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h} u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u^h \left(e_{1s}^h + \sum_j D_{js}\theta_j^h + \tau_{1s}^h, e_{2s}^h + R_s k^h + \tau_{2s}^h\right) \tag{A.26}$$

subject to the budget constraints in the first period

$$c_{10}^h + p_0\left(c_{20}^h + k^h\right) + \sum_{j=1}^J Q_j \theta_j^h \quad \leq \quad e_{10}^h + p_0 e_{20}^h, \tag{A.27}$$

the spot budget constraint in state s

$$\tau_{1s}^h + p_s \tau_{2s}^h \quad = \quad 0, \text{ for } s = 1, \ldots, S, \tag{A.28}$$

the collateral constraint in state s

$$p_s R_s k^h + \sum_j D_{js}\theta_j^h \geq 0, \forall s = 1, \ldots, S, \tag{A.29}$$

and the non-negativity constraint for saving

$$k^h \geq 0, \tag{A.30}$$

- *markets clear for good 1 and good 2 at $t = 0$, for θ_j^h for all $j = 1, \ldots, J$, and for spot trade $\tau_{\ell s}^h$ in state s, respectively:*

$$\sum_h \alpha^h c_{10}^h = \sum_h \alpha^h e_{10}^h, \tag{A.31}$$

$$\sum_h \alpha^h \left(c_{20}^h + k^h\right) = \sum_h \alpha^h e_{20}^h, \tag{A.32}$$

$$\sum_h \alpha^h \theta_j^h = 0, \forall j, \tag{A.33}$$

$$\sum_h \alpha^h \tau_{\ell s}^h = 0, \forall s; \ell = 1, 2. \tag{A.34}$$

C Public Finance Interpretation for the Saving Economy in the Main Text

The budget constraint with the prices of the rights to trade

$$\sum_{p_1} \delta^h (p_1) \left[c_{10}^h (p_1) + p_0 \left[c_{20}^h (p_1) + k^h (p_1)\right] + P_\Delta (p_1) \Delta^h \left(k^h (p_1), p_1\right)\right] \leq e_{10}^h + p_0 e_{20}^h, \tag{A.35}$$

which is the same as (21) in the main text, has a public finance interpretation, as if we were to try to implement the optimum solution by taxes and subsidies. With the constant relative risk aversion (CRRA) utility function, $u^h (c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}$ for $h = 1, 2$, the right to trade in exchange p_1 is

$$\Delta^h \left(k^h, p_1\right) = \left(\frac{\sqrt{p_1}}{1 + \sqrt{p_1}}\right) \left[\sqrt{p_1} \left(e_{21}^h + R k^h\right) - e_{11}^h\right]. \tag{A.36}$$

Substituting the rights to trade (A.36) into the budget constraint for an agent type h (A.35) gives

$$\sum_{p_1} \delta^h (p_1) \left\{c_{10}^h (p_1) + p_0 \left[c_{20}^h (p_1) + k^h (p_1)\right] \leq e_{10}^h + p_0 e_{20}^h - \left[\left(\frac{p_1}{1 + \sqrt{p_1}}\right) P_\Delta (p_1) R\right] k^h \right.$$
$$\left. - \left[\left(\frac{p_1}{1 + \sqrt{p_1}}\right) P_\Delta (p_1)\right] e_2^h + \left[\left(\frac{\sqrt{p_1}}{1 + \sqrt{p_1}}\right) P_\Delta (p_1)\right] e_1^h \right\} \tag{A.37}$$

We can now see that we need to have three types of taxes/subsidies, (i) saving/collateral tax of $\left(\frac{p_1}{1+\sqrt{p_1}}\right) P_\Delta\left(p_1\right) R$ per unit of saving/collateral, k^h, (ii) collateral good endowment tax of $\left(\frac{p_1}{1+\sqrt{p_1}}\right) P_\Delta\left(p_1\right)$ per unit of collateral good 2 endowment at date $t=1$, e_{21}^h, and (iii) subsidy, negative tax $-\left(\frac{\sqrt{p_1}}{1+\sqrt{p_1}}\right) P_\Delta\left(p_1\right)$ per unit of consumption good endowment of good 1 at date $t=1$, e_{11}^h. Endowments matter because they are part of excess demand. The tax/subsidy rate on endowments also depends on the exchange p_1 chosen. That is, the exchange p_1 itself is a choice as far as the household is concerned, so these are not lump sum endowment taxes.[1]

But again we do not need the taxes. We let the markets decide. Markets determine prices, and prices determine allocations.

D The Presence of Rights has No Effect on the Standard Classical Economy

We can show that in a classical economy without pecuniary externalities, the set of competitive equilibrium allocations does not change when segregated markets are introduced. More specifically, start in the extended commodity space with markets for rights at various prices, writing down the programming problem. Then guess that a solution to the first order conditions of the Lagrangian problem (which are both necessary and sufficient) is the solution, quantities and Lagrange multipliers, of the standard classical economy, without externalities, in the standard commodity space. Rights to trade in the extended commodity space are simply again the excess demands of the classical economy, and there are no additional obstacle to trade constraints in either. The guess is verified to be correct. This implies that the (spot) prices in an active segregated exchange must be the same as the shadow prices from the planning problem without the segregated exchanges. That, in turn, ensures that the Lagrange multipliers for the rights constraints for those active spot market exchanges have to be zero. In the analogue decentralized equilibrium, this implies that the prices of the rights to trade in those are zero as well. This makes sense since there is no reason to restrict ("tax" or "subsidize") trade, as that trade is not imposing an externality. However, the Lagrange multipliers or decentralized prices of the rights in inactive exchanges are not necessarily zero. In fact, they should not be zero to help guide agents to choose the optimal exchanges in

[1] This is like looking up marginal rates in a big tax book and settling on which page (or pages) to use, indexed by p_1 that the agent chooses.

equilibrium. This is what is preventing the emergence of new equilibria that might feature some kind of price discrimination.

E Exogenous Incomplete Markets Economy: Geanakoplos and Polemarchakis (1986); Greenwald and Stiglitz (1986)

Consider an exogenously imposed incomplete markets economy. It is an economy with two periods $t = 0, 1$. There are S possible states of nature in the second period, $t = 1$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s, $\sum_s \pi_s = 1$. There are 2 goods, labeled good 1 and good 2, in each date and in each state. There are H types with fractions $\alpha^h > 0$, for $h = 1, 2, ..., H$ such that $\sum_h \alpha^h = 1$. Endowment profiles are e^h_{1s} and e^h_{2s} for goods 1 and 2, where for convenience of notation $s = 0$ is the endowment at date $t = 0$. Preferences of each type h agent are represented by utility u^h.

There are J securities available for purchase or sale in the first period, $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets in the second period $t = 1$ where D_{js} is the payoff of asset j in units of good 1 (the numeraire good) in state $s = 1, 2, \ldots, S$. Here we do not include securities paying in good 2 as there is trade in the two goods in spot markets, so these are not needed. Let θ^h_j denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$ with $\boldsymbol{\theta}^h \equiv \left[\theta^h_j\right]_j$. Here a positive number denotes the purchaser or investor, and negative the issuer, the one making the promise. Let Q_j denote the price of security j with $\mathbf{Q} \equiv [Q_j]_j$. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is, $J < S$. Thus agents type h cannot achieve arbitrary targets for consumption, which enter utility as $u^h \left(e^h_{1s} + \sum_j D_{js}\theta^h_j + \tau^h_{1s}, e^h_{2s} + \tau^h_{2s}\right)$, where τ^h_{1s} and τ^h_{2s} are spot trades in good 1 and 2 in state s, respectively. The u^h are strictly concave with other regularity conditions.

For this exogenous incomplete markets economy, the key set of obstacle-to-trade constraints generating the pecuniary externality are the spot budget constraints

$$C^h_s\left(\tau^h_{\ell s}, \mathbf{p}\right) \equiv \tau^h_{1s} + p_s\tau^h_{2s} = 0, \forall s, h, \tag{A.38}$$

where p_s is the spot market price in state s of good 2 in terms of good 1, and $\mathbf{p} \equiv [p_s]_s$ is the vector of the spot prices. The constraints here are simple spot market budget constraints, but with incomplete markets this is how prices create externalities. To be consistent with the rights to trade $\Delta_s^h(\mathbf{p})$ defined below, we keep the vector of spot prices $\mathbf{p}$ in the general constraint $C_s^h(\tau_{\ell s}^h, \mathbf{p})$. We simplify the notation by restricting ourselves here to two periods, two goods, S states, but it is easy to generalize. Likewise, we can easily incorporate intertemporal savings, for example the storage of good 2, $k^h \geq 0$, as in the numerical example below.

E.1 The Definition of Competitive Equilibrium with Exogenous Incomplete Markets

Definition A.3 (Competitive Equilibrium with Exogenous Incomplete Markets)**.** *A competitive equilibrium is a specification of prices* $(p_0, \mathbf{Q}, \mathbf{p})$, *and an allocation* $(\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h)_h$ *such that*

- *for any agent type* h *as a price taker,* $(\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h)$ *solves*

$$\max_{\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h} u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u^h\left(e_{1s}^h + \sum_j D_{js}\theta_j^h + \tau_{1s}^h, e_{2s}^h + \tau_{2s}^h\right) \tag{A.39}$$

subject to the budget constraints in the first period

$$c_{10}^h + p_0 c_{20}^h + \sum_{j=1}^J Q_j \theta_j^h \quad \leq \quad e_{10}^h + p_0 e_{20}^h, \tag{A.40}$$

and the spot budget constraint in state s

$$\tau_{1s}^h + p_s \tau_{2s}^h \quad = \quad 0, \text{ for } s = 1, \ldots, S; \tag{A.41}$$

- *markets clear for good* $\ell = 1, 2$ *at* $t = 0$*, for* θ_j^h *for all* $j = 1, \ldots, J$*, and for spot trade* $\tau_{\ell s}^h$ *in state* s*, respectively:*

$$\sum_h \alpha^h c_{\ell 0}^h \quad = \quad \sum_h \alpha^h e_{\ell 0}^h, \forall \ell = 1, 2, \tag{A.42}$$

$$\sum_h \alpha^h \theta_j^h \quad = \quad 0, \forall j, \tag{A.43}$$

$$\sum_h \alpha^h \tau_{\ell s}^h \quad = \quad 0, \forall s; \ell = 1, 2. \tag{A.44}$$

The key constraints that generate the externality in this problem are the spot-budget constraints (A.41) for an agent of type h. Note that the spot price p_s is determined by pretrade position of endowments and securities where endowments are exogenous but securities are endogenous, and we write this as $p_s = p_s(\boldsymbol{\theta}, \mathbf{e})$. As in Geanakoplos and Polemarchakis (1986), the dependency generates an indirect price effect from security reallocations. This indirect effect then produces an externality when the security markets are incomplete.

E.2 Source of Inefficiency in the Incomplete Markets Example

Proposition A.1. *The competitive equilibrium with exogenous security markets is (constrained) efficient if and only if the equilibrium allocation* $\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h\right)$ *is first-best optimal or the spot price is independent of security positions, i.e.,* $\frac{\partial p_s}{\partial \theta_j^h} = 0$ *for every state* s*, every security* j *and every agent of type* h*.*

Proof. We begin the proof by deriving the necessary and sufficient conditions for the first-best optimality. The social planner's problem for the first-best optimality is as follows:

Program A.1.

$$\max_{\left(\theta_{i0}^h, \theta_{is}^h\right)_{i,s,h}} u^1\left(e_{10}^1 + \theta_{10}^1, e_{20}^1 + \theta_{20}^1\right) + \beta \sum_s \pi_s u^1\left(e_{1s}^1 + \theta_{1s}^1, e_{2s}^1 + \theta_{2s}^1\right) \tag{A.45}$$

subject to the participation constraints and the resource constraints, respectively,

$$\begin{aligned} u^h\left(e_{10}^h + \theta_{10}^h, e_{20}^h + \theta_{20}^h\right) + \beta \sum_s \pi_s u^h\left(e_{1s}^h + \theta_{1s}^h, e_{2s}^h + \theta_{2s}^h\right) &\geq \overline{\mathscr{U}}^h, for\ h = 2, ..., H, \\ \sum_h \alpha^h \theta_{is}^h &= 0, for\ i = 1, 2; s = 0, 1, ..., S \end{aligned}$$

Lemma A.1. *The necessary and sufficient conditions for the first-best optimality are as follows:*

$$\frac{\gamma_u^h u_{i0}^h}{\alpha^h} = \frac{\gamma_u^{\tilde{h}} u_{i0}^{\tilde{h}}}{\alpha^{\tilde{h}}}, \forall h, \tilde{h} = 1, ..., H; i = 1, 2 \tag{A.46}$$

$$\frac{\gamma_u^h \beta \pi_s u_{is}^h}{\alpha^h} = \frac{\gamma_u^{\tilde{h}} \beta \pi_s u_{is}^{\tilde{h}}}{\alpha^{\tilde{h}}}, \forall h, \tilde{h} = 1, ..., H; i = 1, 2, s = 1, ..., S, \tag{A.47}$$

where γ_u^h *is the Lagrange multipliers for the participation constraints for* h *(normalized by setting* $\gamma_u^1 = 1$*) and* $u_{is}^h = \frac{\partial u^h}{\partial c_{is}^h}$ *is the marginal utility of an agent of type* h *with respect to* c_{is}*.*

We now consider the following social planner's problem for the economy with exogenous security markets.

Program A.2.

$$\max_{\left(\theta_{10}^h, \theta_{20}^h, \theta_j^h, \tau_{1s}^h, \tau_{2s}^h\right)_h} u^1\left(e_{10}^1 + \theta_{10}^1, e_{20}^1 + \theta_{20}^1\right) + \beta \sum_s \pi_s u^1\left(e_{1s}^1 + \sum_j D_{js}\theta_j^h + \tau_{1s}^1, e_{2s}^1 + \tau_{2s}^1\right) \tag{A.48}$$

subject to the participation constraints, the resource constraints, and the obstacle-to-trade constraints, respectively,

$$u^h\left(e^h_{10}+\theta^h_{10}, e^h_{20}+\theta^h_{20}\right)+\beta\sum_s \pi_s u^h\left(e^h_{1s}+\sum_j D_{js}\theta^h_j+\tau^h_{1s}, e^h_{2s}+\tau^h_{2s}\right)\geq \overline{\mathscr{U}}^h, \forall h, \tag{A.49}$$

$$\sum_h \alpha^h\theta^h_{i0}=0, \forall i, \tag{A.50}$$

$$\sum_h \alpha^h\theta^h_j=0, \forall j, \tag{A.51}$$

$$\sum_h \alpha^h\tau^h_{1s}=0, \forall s, \tag{A.52}$$

$$\tau^h_{1s}+p_s\left(\theta^1_s, ..., \theta^H_s\right)\tau^h_{2s}=0, \forall s, h. \tag{A.53}$$

Note that the resource (market-clearing) constraints for τ^h_{2s} are omitted due to Walras law. A solution to this social planner's problem is called a constrained optimal allocation.

The first order conditions for $\theta^h_{10}, \theta^h_{20}, \tau^h_{1s}, \tau^h_{2s}, \theta^h_j$ are as follows:

$$\gamma^h_u\beta\pi_s u^h_{10}+\alpha^h\mu^\theta_{10} = 0, \tag{A.54}$$

$$\gamma^h_u\beta\pi_s u^h_{20}+\alpha^h\mu^\theta_{20} = 0, \tag{A.55}$$

$$\gamma^h_u\beta\pi_s u^h_{1s}+\alpha^h\mu^\tau_{1s}+\gamma^h_s = 0, \forall s=1, ..., S, \tag{A.56}$$

$$\gamma^h_u\beta\pi_s u^h_{2s}+p_s\gamma^h_s = 0, \forall s=1, ..., S, \tag{A.57}$$

$$\gamma^h_u\beta\sum_s\pi_s u^h_{1s}D_{js}+\alpha^h\mu^\theta_j+\sum_s\frac{\partial p_s}{\partial\theta^h_j}\sum_{\tilde{h}}\gamma^{\tilde{h}}_s\tau^{\tilde{h}}_{2s} = 0, \forall j=1, ..., J, \tag{A.58}$$

where $\gamma^h_s, \gamma^h_s, \mu^\tau_s, \mu^\theta_j$ are the Lagrange multipliers for the obstacle to trade or spot-market constraints in state s, for the participation constraints for h (normalize by setting $\gamma^1_u=1$), for the resource constraints for τ^h_{1s}, and for the resource constraints for θ^h_j. Note that $u^h_{is}=\frac{\partial u^h}{\partial c^h_{is}}$.

The proof is divided into two parts as follows:

(i) (⇐) We now show that an allocation that satisfies the necessary and sufficient conditions for the first-best optimality (A.46)-(A.47) must satisfies the first order conditions (A.54)-(A.58). It is not difficult to see that this will be the case if the externality term, the last term of (A.58), is vanished, i.e.,

$$\sum_s\frac{\partial p_s}{\partial\theta^h_j}\sum_{\tilde{h}}\gamma^{\tilde{h}}_s\tau^{\tilde{h}}_{2s}=0 \tag{A.59}$$

It is obvious that if the spot price is independent of security positions, i.e., $\frac{\partial p_s}{\partial \theta_j^h} = 0$ for every state s, every security j and every agent of type h, then condition (A.59) holds.

We now need to show that if the constrained optimal allocation is first-best optimal, then the no-externality condition (A.59) must hold. Since the allocation is first- best optimal, it must satisfies conditions (A.46) and (A.47), which imply that $\left(\frac{\gamma_s^{\tilde{h}}}{\alpha^{\tilde{h}}}\right)$ must be constant across agents, i..e, for each s

$$\frac{\gamma_s^h}{\alpha^h} = \frac{\gamma_s^{\tilde{h}}}{\alpha^{\tilde{h}}} = \mathit{\Gamma}_s, \forall h, \tilde{h}. \tag{A.60}$$

Using these conditions, we can then show that

$$\begin{aligned} \sum_s \frac{\partial p_s}{\partial \theta_j^h} \sum_{\tilde{h}} \gamma_s^{\tilde{h}} \tau_{2s}^{\tilde{h}} &= \sum_s \frac{\partial p_s}{\partial \theta_j^h} \sum_{\tilde{h}} \left(\frac{\gamma_s^{\tilde{h}}}{\alpha^{\tilde{h}}}\right) \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} && \text{(A.61)} \\ &= \sum_s \frac{\partial p_s}{\partial \theta_j^h} \sum_{\tilde{h}} \mathit{\Gamma}_s \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} = \sum_s \frac{\partial p_s}{\partial \theta_j^h} \mathit{\Gamma}_s \sum_{\tilde{h}} \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} = 0, && \text{(A.62)} \end{aligned}$$

where the last equation results from the resource constraints for τ_{2s}^h. This proves that the no-externality condition (A.59) holds. To sum up, we prove that there is no externality *if* the constrained optimal allocation is first-best optimal or the spot price is independent of security positions, i.e., $\frac{\partial p_s}{\partial \theta_j^h} = 0$ for every state s, every security j and every agent of type h.

(ii) ($\Rightarrow$) Unfortunately, we cannot generally prove the reversed statement but, as shown in Geanakoplos and Polemarchakis (1986), it is true generically (it is true except for some unlikely cases). The key idea is that the indirect price effects could be canceling each other out only if the equilibrium allocation is first-best optimal in most cases. But this does not happen generally.

□

E.3 Remedy for the Externality in the Incomplete Markets Economy

As already noted, a key step defines type h's rights to trade $\Delta_s^h(\mathbf{p})$ in the spot markets at prices p_s. Type h chooses both the amount of these rights to trade, that is, the trades at p_s, and the vector $\mathbf{p} = (p_s)_{s=1}^S$ itself. To repeat, there is in effect a market place exchange indexed by prices $\mathbf{p}$ where security trades will be entered into and priced at $t = 0$ and where goods will be exchanged in spot markets at state s at the same price p_s. For these rights to have meaning these exchanges must be segregated and choices of the agents must be exclusive.

In more detail, the quantity of rights purchased over states $s = 1, \ldots, S$ is a vector $\boldsymbol{\Delta}^h(\mathbf{p}) = \left[\Delta_s^h(\mathbf{p})\right]_{s=1}^S$. In a particular state s, $\Delta_s^h(\mathbf{p})$ is defined as the standard excess demand for the numeraire, good 1, of an agent type h in spot markets in state s. Namely, $\Delta_s^h(p_1, \ldots, p_s, \ldots, p_S) = \tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)$ is the solution to the state s utility maximization problem at price p_s:

$$\left(\tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right), \tau_{2s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)\right) = \underset{\tau_{1s}^h, \tau_{2s}^h}{\operatorname{argmax}}\, u^h\left(e_{1s}^h + \sum_{j=1}^J D_{js}\theta_j^h + \tau_{1s}^h, e_{2s}^h + \tau_{2s}^h\right) \tag{A.63}$$

subject to the spot-budget constraints (A.38). This choice of rights could be costly to buy or alternatively it could generate revenue. In particular, let $P_\Delta(\mathbf{p}, s)$ denote the market price of rights to spot trade in exchange $\mathbf{p}$ in state s, with components as desired spot prices and the vector running over all states s. Then the net cost is this per unit price times the quantity of rights $\Delta_s^h(\mathbf{p})$ just defined. Namely, $\sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})$ if $\mathbf{p}$ is chosen. Let $\delta^h(\mathbf{p})$ be the indicator variable which is equal to 1 for the chosen $\mathbf{p}$ and is zero otherwise. Thus the budget term will be $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})$.

The key tie-in is that security trades $\boldsymbol{\theta}^h(\mathbf{p})$ are also tied to the choice of exchange $\mathbf{p}$. That is, let $Q_j(\mathbf{p})$ denote the price of security j traded in exchange $\mathbf{p}$. This then has a net cost in the budget $\sum_{\mathbf{p}} \delta^h(\mathbf{p}) \sum_j Q_j(\mathbf{p})\theta_j^h(\mathbf{p})$. Both costs of rights to trade and the tie-in to securities are subtracted from the value of endowments at $t = 0$ leaving consumption as a residual. The entire budget is the following

$$\sum_{\mathbf{p}} \delta^h(\mathbf{p})\left[c_{10}^h + p_0 c_{20}^h + \sum_j Q_j(\mathbf{p})\theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\Delta_s^h(\mathbf{p})\right] \leq e_{10}^h + p_0 e_{20}^h. \tag{A.64}$$

Finally, the spot prices $\mathbf{p}$ and security prices $Q_j(\mathbf{p})$ will have to be such as to attain market clearing in rights to trade:

$$\sum_h \delta^h(\mathbf{p})\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}, \tag{A.65}$$

and market clearing in securities

$$\sum_h \delta^h(\mathbf{p})\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p}. \tag{A.66}$$

Also the spot market in each state s in exchange $\mathbf{p}$ must be cleared, consistent with the agent types who have chose to trade there, validating their choice of $\mathbf{p}$.

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau^h_{\ell s}(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = 1,2. \tag{A.67}$$

Note that due to the maximization of (A.63) subject to (A.38) that the chosen τ^h_{1s} at $\mathbf{p}$ that appear in (A.67) will be the τ^{h*}_{1s} in (A.63), in turn equal to the rights $\Delta^h_s(\mathbf{p})$ purchased. Finally, equations (A.65) can be satisfied trivially for inactive exchanges where $\delta^h(\mathbf{p}) = 0$ for all h.

In general, spot prices p_s can be a complex mapping from pre-trade endowments and security holdings. There are few a priori restrictions on individual and aggregate excess demands. But, conceptually, for the individual this does not matter, as all she cares about are the chosen prices at which she will trade and the associated implication for rights, security, and spot trades. Finding an equilibrium is the economist's problem, not the agent's problem.

Let $\mathbf{x}^h(\mathbf{p}) = \left(\mathbf{c}^h_0, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right)$ denote a typical bundle or allocation for an agent type h, where again $\boldsymbol{\Delta}^h(\mathbf{p}) \equiv \left[\Delta^h_s(p_s)\right]_s$. If $\delta^h(\mathbf{p}) = 0$, then the rest of the $\mathbf{p}$ contingent choices need not be specified, as agent h is choosing not to trade at $\mathbf{p}$.

Definition A.4. *A competitive equilibrium with segregated exchanges indexed by* $\mathbf{p}$ *is a specification of allocation* $\left[\mathbf{x}^h(\mathbf{p})\right]_{h,\mathbf{p}} \equiv \left[\mathbf{c}^h_0, \delta^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right]_{h,\mathbf{p}}$ *and prices* $(p_0, \mathbf{Q}, \mathbf{p}, \mathbf{P}_\Delta)$ *such that*

(i) for any agent type h as a price taker, $\left[\mathbf{x}^h(\mathbf{p})\right]_\mathbf{p}$ *solves*

$$\max_{\left[x^h(\mathbf{p})\right]_\mathbf{p}} \sum_\mathbf{p} \delta^h(\mathbf{p}) \left[u\left(c^h_{10}, c^h_{20}\right) + \sum_s \pi_s u\left(e^h_{1s} + \sum_j D_{js}\theta^h_j(\mathbf{p}) + \tau^h_{1s}(\mathbf{p}), e^h_{2s} + \tau^h_{2s}(\mathbf{p})\right)\right]$$

subject to the budget constraints in the first period

$$\sum_\mathbf{p} \delta^h(\mathbf{p}) \left[c^h_{10} + p_0 c^h_{20} + \sum_j Q_j(\mathbf{p})\,\theta^h_j(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s)\,\Delta^h_s(\mathbf{p})\right] \le e^h_{10} + p_0 e^h_{20},$$

and the spot-budget constraint in state s

$$\sum_\mathbf{p} \delta^h(\mathbf{p}) \left[\tau^h_{1s}(\mathbf{p}) + p_s \tau^h_{2s}(\mathbf{p})\right] = 0, \forall s,$$

(ii) markets clear for good ℓ in $t = 0$, for securities j paying good 1, for good ℓ in state s, and

for rights to trade in exchange $\mathbf{p}$ *for state* s*, respectively,*

$$\sum_h \alpha^h c_{10}^h = \sum_h \alpha^h e_{10}^h,$$

$$\sum_h \alpha^h c_{20}^h = \sum_h \alpha^h e_{20}^h,$$

$$\sum_h \delta^h (\mathbf{p}) \alpha^h \theta_j^h (\mathbf{p}) = 0, \forall j; \mathbf{p},$$

$$\sum_h \delta^h (\mathbf{p}) \alpha^h \tau_{\ell s}^h (\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = 1, 2,$$

$$\sum_h \delta^h (\mathbf{p}) \alpha^h \Delta_s^h (\mathbf{p}) = 0, \forall s; \mathbf{p}.$$

For the collateral economy, there is of course the additional constraints (**??**). The collateral economy is also special in that in addition to these collateral constraints, the security structure is complete, that is, the securities, θ_{1s}^h and θ_{2s}^h, are the Arrow-Debrue securities, the selection $\delta^h (\mathbf{z})$ and indexation of securities, spot trades $\tau_{\ell s}^h (z_s)$, the rights $\Delta_s^h (z_s)$ and prices $p (z_s)$ can all be written in terms of the fundamental ratio z_s, and everything is state contingent (no need for vectors).

E.4 Illustrative Example for the Exogenous Incomplete Markets Economy

Consider an economy with two periods, $t = 0, 1$. There are two states $s = 1, 2$ in the second period. Let the probability of state s is $\pi_s = \frac{1}{2}$. For simplicity, there is only one good in the first period $t = 0$, while there are two goods $\ell = 1, 2$ in the second period $t = 1$. Each unit of storage of the single good in the first period becomes one unit of good 2 in the second period regardless of the state (no shocks on the return to storage). Notationally, k units of storage in $t = 0$ gives k units of good 2 in every state in the second period.

There are two types of agents, $H = 2$, both of which have an identical logarithmic utility function

$$u^h(c) = \ln c, \tag{A.68}$$

which is homothetic. As a result, the equilibrium spot price p_s is determined by the ratio of the commodity aggregates, as in the collateral economy. In particular, with the log utility, $p_s = z_s$. Note that the externality exists in this economy due to the interaction between the incompleteness of the markets and storage, as first period decisions impact the second period price. Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor is $\beta = 1$.

The endowment profiles of the agents are shown in Table A.1 below. Note that all risk is idiosyncratic, with type 1 relatively well endowed in state $s = 1$ and vice versa for type 2. Note also that the symmetry of the endowments and the homogeneity of the preferences imply that an equilibrium allocation is symmetric. In addition, the endowment is structured in such a way that both types would like to save ($k^h > 0$).

Table A.1: Endowment profiles of the agents.

	e^h_{10}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	10	5	5	1	1
$h = 2$	10	1	1	5	5

Definition A.5 (Competitive Equilibrium with Saving-Borrowing Only)**.** *A competitive equilibrium is a specification of prices* (p_1, p_2)*, and an allocation* $\left(c^h_{10}, c^h_{\ell s}, k^h\right)_h$ *such that*

- *for any agent type* h *as a price taker,* $\left(c^h_{10}, c^h_{\ell s}, k^h\right)$ *solves*

$$\max_{c^h_{10}, c^h_{\ell s}, k^h} u\left(c^h_{10}\right) + \beta \sum_s \pi_s \left[u\left(c^h_{1s}\right) + u\left(c^h_{2s}\right)\right] \tag{A.69}$$

subject to the budget constraints in period $t = 0$ *and state* $s = 1, 2$*, respectively*

$$c^h_{10} + k^h = e^h_{10} \tag{A.70}$$

$$c^h_{1s} + p_s c^h_{2s} = e^h_{1s} + p_s \left[e^h_{2s} + k^h\right], \forall s. \tag{A.71}$$

- *markets clear for good 1 at* $t = 0$*:*

$$\sum_h \alpha^h \left(c^h_{10} + k^h\right) = \sum_h \alpha^h e^h_{10}, \tag{A.72}$$

for good 1 in each state $s = 1, 2$*:*

$$\sum_h \alpha^h c^h_{1s} = \sum_h \alpha^h e^h_{1s}, \tag{A.73}$$

and for good 2 in each state $s = 1, 2$*:*

$$\sum_h \alpha^h c^h_{2s} = \sum_h \alpha^h \left(c^h_{2s} + k^h\right), \tag{A.74}$$

With a homogeneous homothetic utility function, the spot price in each state s is determined by the ratio of the aggregate resources:

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{\sum_h \alpha^h \left(e_{2s}^h + k^h\right)}, \tag{A.75}$$

with the equilibrium spot price function $p_s = p(z_s) = z_s$.

With symmetry, the first best allocation ("fb") with state contingent transfers has both agents save $k_0^{fb} = 3.5$, which implies that the equilibrium spot price of good 2 $p_s^{fb} = 0.4615$ in all states. On the other hand, the competitive equilibrium with externality for Environment E.4 is numerically solved and presented in columns 1 and 2 of Table A.2. With externality, each agent type saves more $k^{ex} = 4.3077$ to try to cover some of the idiosyncratic risk, which leads to a lower equilibrium spot price $p_s^{ex} = 0.4105$ in all states.

Table A.2: Competitive equilibrium with incomplete markets and the corresponding constrained optimal solutions.

	equilibrium with externality		constrained optimality		equilibrium with rights to trade	
	$h=1$	$h=2$	$h=1$	$h=2$	$h=1$	$h=2$
c_{10}^h	5.6923	5.6923	5.9767	5.9767	5.9767	5.9767
k^h	4.3077	4.3077	4.0233	4.0233	4.0233	4.0233
c_{11}^h	4.3077	10.7437	4.4272	10.3643	4.4272	10.3643
c_{12}^h	1.5895	3.8718	1.5728	3.6822	1.5728	3.6822
c_{21}^h	10.7437	4.3077	10.3643	4.4272	10.3643	4.4272
c_{22}^h	3.8718	1.5895	3.6822	1.5728	3.6822	1.5728
p_s	0.4105	0.4105	0.4272	0.4272	0.4272	0.4272
Expected Utility	4.5768	4.5768	4.5791	4.5791	4.5791	4.5791

We now solve for the competitive equilibrium with rights to trade in segregated exchanges using the following Pareto program with equal Pareto weights, $\lambda^h = \frac{1}{2}$ for all $h = 1, 2$. Let $\mathbf{x}^h \equiv \left[x^h\left(c_{10}, k, \mathbf{p}, \mathbf{\Delta}\right)\right]$ be a typical lottery for an agent of type h. We again impose on the grid of the lottery that a positive mass can be put on a grid with the following property only: for each bundle $(c_{10}, k, \mathbf{p}, \mathbf{\Delta})$ and each agent type h,

$$\Delta_s(\mathbf{p}) = p_s\left(e^h_{2s} + k\right) - e^h_{1s}, \tag{A.76}$$

The Pareto program with rights to trade is

$$\max_{\mathbf{x}^h} \sum_h \lambda^h \alpha^h \sum_{c_{10},k,\mathbf{p},\mathbf{\Delta}} x^h(c_{10}, k, \mathbf{p}, \mathbf{\Delta}) \left\{ u(c_{10}) + \beta \sum_{s=1}^{S} \pi_s V\left(e^h_{1s,}, e^h_{2s} + k; p_s\right) \right\} \tag{A.77}$$

subject to

$$\sum_{c_{10},k,\mathbf{p},\mathbf{\Delta}} x^h(c_{10}, k, \mathbf{p}, \mathbf{\Delta}) = 1, \forall h; \tag{A.78}$$

$$\sum_h \alpha^h \sum_{c_{10},k,\mathbf{p},\mathbf{\Delta}} x^h(c_{10}, k, \mathbf{p}, \mathbf{\Delta}) \left\{ c_{10} + k - e^h_{10} \right\} = 0; \tag{A.79}$$

$$\sum_h \alpha^h \sum_{c_{10},k,\mathbf{\Delta}} x^h(c_{10}, k, \mathbf{p}, \mathbf{\Delta}) \Delta_s = 0, \forall s; \mathbf{p} \tag{A.80}$$

where the indirect utility function for an agent type h in state s is defined by

$$V\left(e^h_{1s,}, e^h_{2s} + k; p_s\right) = \max_{\tau_{1s},\tau_{2s}} u\left(e^h_{1s} + \tau_{1s}\right) + u\left(e^h_{2s} + k + \tau_{2s}\right) \tag{A.81}$$

subject to the spot market budget constraint

$$\tau_{1s} + p_s \tau_{2s} = 0 \tag{A.82}$$

According to the second welfare theorem, the constrained optimal allocation can be decentralized as the competitive equilibrium with rights to trade in segregated exchanges. In fact, we numerically solve the liner programming problem above in Matlab and then recover the equilibrium prices using part of the proof of the theorem. The outcome for Environment E.4 is presented in the columns 5 and 6 of Table A.2.

The equilibrium allocation with rights to trade in segregated exchanges has only *one active spot market* $\mathbf{p}^{op}$ with $p^{op}_s = 0.4272$ for all $s = 1, 2$. The equilibrium savings here is lower than the one in competitive equilibrium with the externality, i.e., $k^{op} = 4.0233$ (but still higher than in the first best). Table A.3 presents equilibrium prices/fees of rights to trade in exchange vector $\mathbf{p} = (p_1, p_2)$ in each state $s = 1, 2$, the $P_\Delta(\mathbf{p}, s)$ with argument ranging over $\mathbf{p}$ and s that is, including over inactive exchanges. Note that the prices of the rights to trade with the spot price p_s in different exchanges are clearly different, i.e., $P_\Delta((p_1, p_2), 1) \neq P_\Delta((p_1, \tilde{p}_2), 1)$ when $p_2 \neq \tilde{p}_2$. Note that here the vector is different in the second component, yet this makes the rights price for trading in the first state different.

Table A.3: The equilibrium price of the rights to trade in exchange $\mathbf{p} = (p_1, p_2)$ in each state s, $P_\Delta(\mathbf{p}, s)$.

$\mathbf{p}$		$P_\Delta(\mathbf{p}, 1)$	$P_\Delta(\mathbf{p}, 2)$
p_1	p_2		
0.41609	0.41609	0.07456	0.07456
0.41609	0.42715	0.07807	0.09456
0.42715	0.41609	0.09455	0.07806
0.42715	**0.42715**	**0.13342**	**0.13342**
0.42715	0.46154	0.17094	0.23213
0.46154	0.42715	0.23213	0.17094

In this equilibrium, each agent type buys/sells the rights to trade in an exchange $\mathbf{p}^{op} = (0.42715, 0.42715)$. Due to symmetry, an agent type $h = 1$ sells the rights $\Delta^1(\mathbf{p}^{op}, 1) = -1.1457$ in state $s = 1$, the numeraire good 1 and hence buys good 2 in $s = 1$. Agent type $h = 1$ buys the same amount of good 1 in state $s = 2$, $\Delta^1(\mathbf{p}^{op}, 2) = 1.1457$, and hence agent type 1 sells good 2 at $s = 2$. This is crucial as savings of type 1 is motivated by the shortfall of type 1's endowment in state $s = 2$. That is, this is where the exposure to idiosyncratic risk for agent type 1 is doing damage, bringing too much good 2 to the second period, creating the externality. The markets for rights to trade in good 1 can remove the externality through its marginal impact on the decision to save of each agent type. In effect, each pays a "tax" when selling good 2 and buying good 1 in the state which motivated the saving. The situation is reversed for agent type 2 in terms of the ordering over goods and states, but the same in terms of saving. In total, the net trade in the rights to trade will be zero in net value for both agent types at $t = 0$. What agent 1 buys agent 2 sells and vice versa, and each trade has the same value. That is, each agent does actively trade rights but with no implication for wealth at $t = 0$. The key is the marginal impact of each rights to trade on saving decisions. Note that the optimal lottery is degenerated; that is, all agents of each type choose the only one exchange as the probability measure at the optimal allocation $x^h(\mathbf{w}^{op}) = 1.000$ for all $h = 1, 2$, as shown in Table A.4. In fact, it is identical to the solution of the planning problem below.

Constrained Planning Problem

Here the constrained optimal allocation can be computed by maximizing the expected utility of the $t = 0$ representative consumer, exploiting the symmetry, subject to spot market constraints with price p_s replaced by the appropriate clearing ratio of commodities. The solution to the following planning problem below is confirmed to be the solution to the equilibrium with rights. For notational purposes, we drop the type-specific index h.

$$\max_{c_{10},c_{\ell s},k} u\left(c_{10}\right)+\beta \sum_{s} \pi_s\left[u\left(c_{1s}\right)+u\left(c_{2s}\right)\right] \tag{A.83}$$

subject to

$$c_{10}+k=e_{10}, \tag{A.84}$$

$$c_{1s}+\left(\frac{E_{1s}}{E_{2s}+k}\right) c_{2s}=e_{1s}+\left(\frac{E_{1s}}{E_{2s}+k}\right)\left(e_{2s}+k\right), \forall s=1,2, \tag{A.85}$$

where the aggregate of good ℓ in state s is $E_{\ell s}=\sum_h \alpha^h e_{\ell s}^h$. We here write the spot budget constraint in terms of the ratio of commodity aggregates. The optimal allocation is numerically solved and presented in columns 3 and 4 of Table A.2. In fact, the necessary condition for the optimality can be formulated as a cubic function, which leads to the only one feasible (as a real number) solution.

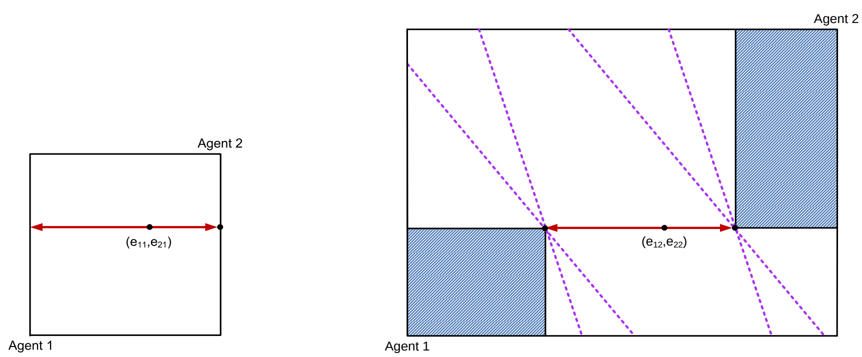
Table A.4: The equilibrium allocation.

	$h=1$	$h=2$
c_{10}^{op}	5.9767	5.9767
k^{op}	4.0233	4.0233
p_1^{op}	0.42715	0.42715
p_2^{op}	0.42715	0.42715
Δ_1^{op}	-1.1457	1.1457
Δ_2^{op}	1.1457	-1.1457
x^h	1.000	1.000
$P_\Delta\left(\mathbf{p}^{op},1\right)$	0.13342	0.13342
$P_\Delta\left(\mathbf{p}^{op},2\right)$	0.13342	0.13342

E.5 Markets for Rights to Trade Do Not Complete the Securities Markets

Of course, one might wonder if our method solves the externality problem by simply completing the markets? By allowing agents to choose markets with pre-specified spot prices in each state s, we effectively create state-contingent transfers of wealth at least to some degree. But is it enough to achieve the first best allocation? The answer is generally, no. Exogenous incomplete markets and the positivity of spot prices still restrict how much wealth transfers we can make in each state. Technically, the feasible set with incomplete markets and rights to trade is generically a strict subset of the feasible set with the complete markets.

See Figure 1 for an illustrative example. This example assumes the stereotypical debt contract, a bond that pays the same amount of good 1 in each two states. However, in state $s = 2$, there is more of good 1 and good 2 overall. Then, no matter what the price ratio p_s in state $s = 2$, certain regions cannot be reached. The main point is that the scarcity in state $s = 1$ can affect the feasibility in state $s = 2$ because the markets are incomplete.



(a) A feasible set in state $s = 1$.

(b) A feasible set in state $s = 2$. The shaded areas are not feasible.

Figure 1: Feasible Sets in state $s = 1$ and $s = 2$ when markets are incomplete. The only available security pays the same amount of good 1 in both states.

F The Collateral Economy: Kilenthong and Townsend (2014)

This section presents the additional results, proofs and numerical examples regarding the collateral economy, which is closely related to the leading example in the text. Given that the preferences are assumed to be homogeneous and homothetic, an equilibrium spot price in state s, p_s, is uniquely determined by the market fundamental, the ratio between good 1 and good 2, z_s; that is, $p_s = p(z_s)$. As a result, an exchange can be defined by either the spot price p_s or equivalently by the market fundamental z_s. We therefore use them interchangeably.

F.1 Competitive Collateral Equilibrium (with an externality)

Agents can trade in spot markets, and let $\tau_{\ell s}^h$ be the amount of good $\ell = 1, 2$ bought by an agent type h in the spot markets at state s at prices p_s, respectively. Let p_0 be the price of good 2 in period $t = 0$, and $Q_{\ell s}$ be the prices of securities in the ex ante $t = 0$ market paying in good $\ell = 1, 2$ in state s, respectively, all priced in the numeraire good 1 at $t = 0$. For notational convenience, let $\mathbf{p} \equiv (p_1, \ldots, p_S)$ and $\mathbf{Q} \equiv [Q_{\ell s}]_{\ell,s}$. A collateral equilibrium is thus defined:

Definition A.6. *A competitive collateral equilibrium is a specification of prices* $(p_0, \mathbf{Q}, \mathbf{p})$*, and an allocation* $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ *such that*

(i) for any agent type h *as a price taker,* $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ *solves*

$$\max_{\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)} u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s}^h + \tau_{1s}^h, e_{2s}^h + R_s k^h + \theta_{2s}^h + \tau_{2s}^h\right) \tag{A.86}$$

subject to the collateral constraint for each state s*:*

$$p_s R_s k^h + \theta_{1s}^h + p_s \theta_{2s}^h \geq 0, \ \forall s, \tag{A.87}$$

the budget constraint at $t = 0$*:*

$$c_{10}^h + p_0\left(c_{20}^h + k^h\right) + \sum_s Q_{1s}\theta_{1s}^h + \sum_s Q_{2s}\theta_{2s}^h \leq e_{10}^h + p_0 e_{20}^h, \tag{A.88}$$

and the spot budget constraint at each state s*:*

$$\tau_{1s}^h + p_s \tau_{2s}^h = 0, \tag{A.89}$$

(ii) markets clear for good 1 at $t = 0$, for good 2 at $t = 0$, for $\theta^h_{\ell s}$ in state s, and for $\tau^h_{\ell s}$ in state s, respectively:

$$\sum_h \alpha^h c^h_{10} \leq \sum_h \alpha^h e^h_{10}, \tag{A.90}$$

$$\sum_h \alpha^h \left[c^h_{20} + k^h\right] \leq \sum_h \alpha^h e^h_{20}, \tag{A.91}$$

$$\sum_h \alpha^h \theta^h_{\ell s} = 0, \ \forall s; \ell = 1, 2; \tag{A.92}$$

$$\sum_h \alpha^h \tau^h_{\ell s} = 0, \ \forall s; \ell = 1, 2; \tag{A.93}$$

The necessary maximizing condition for a collateral equilibrium (ce) related to collateral allocation k^h (an interior solution to the consumer problem) is given by, for any h,

$$p_0 = \frac{u^h_{20}}{u^h_{10}}\Big|_{ce} = \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\gamma^h_{cc-s}}{u^h_{10}} p_s R_s, \tag{A.94}$$

where $u^h_{\ell 0} = \frac{\partial u\left(c^h_{10}, c^h_{20}\right)}{\partial c_{\ell 0}}$, $u^h_{\ell s} = \frac{\partial u\left(c^h_{1s}, c^h_{2s}\right)}{\partial c_{\ell s}}$ for $\ell = 1, 2$, and γ^h_{cc-s} is the Lagrange multiplier for the collateral constraint (A.87) in state s for an agent type h.

F.2 Collateral Constrained Optimality

Attainable allocations are those that can be achieved by exchanges of securities and collateral in date $t = 0$ as well as exchanges of consumption goods in date $t = 1$ at state s, but respecting that agents can trade good 1 for good 2 freely in each state s, and the planner cannot prevent it[2]. In other words, a planner can only reallocate goods with the same instruments as the agents, the security holdings and promises, and must respect the possibility of active spot market trade. Thus the planner must respect the associated collateral constraints assigning collateral to promises but allowing collateral to be unwound at the implicit or explicit spot price inherent in the proposed allocation[3]. Accordingly, attainable allocations are defined using the spot-price function $p(z_s)$. Likewise Geanakoplos and Polemarchakis (1986) search for a Pareto improving allocation of assets recognizing the influence of an asset allocation on spot prices, that spot prices are determined by the conditions that aggregate excess demands must equal zero.[4]

[2] An individual agent, in contrast, has zero mass and no influence on prices regardless of the market structure.

[3] Or using a forward price ratio to value promises made in good 1 in terms of good 2, as in footnote **??**.

[4] See Section 7.2, page 87 in that paper.

Definition A.7. *An allocation* $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ *is* attainable *if*

(i) it satisfies resource constraints (A.90)-(A.93);

(ii) for each agent type h, it satisfies the collateral constraints for each state s:

$$p(z_s)R_s k^h + \theta_{1s}^h + p(z_s)\theta_{2s}^h \quad \geq \quad 0, \ \forall s, \tag{A.95}$$

and the type h spot budget constraints

$$\tau_{1s}^h + p\left(z_s\right)\tau_{2s}^h = 0, \forall s; \tag{A.96}$$

(iii) the consistency constraint

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s \sum_h \alpha^h k^h + \sum_h \alpha^h e_{2s}^h} \tag{A.97}$$

holds for all s.

A constrained optimal allocation is characterized using the following planner's problem. Let $\overline{U}^h$ be the reservation utility level for an agent type h.

Program A.3. *The Pareto Program with collateral constraints:*

$$\max_{\left(\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h, z_s\right)} \quad u\left(c_{10}^1, c_{20}^1\right) + \beta \sum_s \pi_s u\left(e_{1s}^1 + \theta_{1s}^1 + \tau_{1s}^1, e_{2s}^1 + R_s k^1 + \theta_{2s}^1 + \tau_{2s}^1\right) \tag{A.98}$$

subject to (A.90)-(A.93), (A.95)-(A.96), (A.97) and the participation constraint for each $h = 2, \cdots, H$,

$$u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s}^h + \tau_{1s}^h, e_{2s}^h + R_s k^h + \theta_{2s}^h + \tau_{2s}^h\right) \quad \geq \quad \overline{U}^h, \tag{A.99}$$

and non-negativity constraints for consumption and collateral allocations.

As is typically the case, it suffices to consider only equal-treatment-of-equals in the Pareto problem[5]. Let μ_{cc-s}^h, and μ_u^h denote the Lagrange multipliers for the collateral constraint (A.95)

[5] Again, for exposition simplicity and without any real loss, we consider only equal-treatment (for each type), and interior solutions (i.e., the non-negativity constraint for k^h is neglected). With homothetic and strictly concave preferences, and no non-convexity, agents of the same type will optimally choose the same allocation in an equilibrium; that is, given the same market prices in equilibrium. Thus, a collateral equilibrium allocation has equal treatment of equals property. More generally, externalities in this class of models, if they exist, have nothing to do with the equal treatment of equals property.

for agent h in state s, and for the participation constraint (A.99) for agent h, respectively. For notational convenience, let $\mu^1_{\bar{u}} = 1$. A necessary condition[6] for constrained optimality (op) related to collateral allocation k^h is given by, for any h,

$$\left.\frac{u^h_{20}}{u^h_{10}}\right|_{op} = \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s}, \tag{A.100}$$

where $p'(z_s) = \frac{\partial p(z_s)}{\partial z_s}$, $K = \sum_h \alpha^h k^h$.

The Externality

Note that an infinitesimal agent of type h takes a spot price, $p(z_s)$, as invariant to his or her own actions in the collateral equilibrium. To the contrary, the constrained planner can influence the spot prices $p(z_s)$ through collateral assignments for the agents of type $h = 1, 2, \ldots, H$, in period $t = 0$, namely k^h, which affect in turn the market fundamentals z_s. This key influence is the term in $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K}$. If the very last term in (A.100) were zero and we set $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}}$, then condition (A.94) is exactly the same as (A.100), and there would be no externality. The last term in (A.100) could be zero if either $\mu^{\tilde{h}}_{cc-s} = 0$, that is, no collateral constraint is binding for any $\tilde{h}$ or $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$.[7]

F.3 Constrained Non-Optimality of The Collateral Economy

Proposition A.2. *With continuous, strictly concave, strictly increasing, and identical homothetic utility functions, a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e.* $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ *for all* h *and all* s.

[6]Given that the constraint set is not convex, this optimality condition is necessary but may not be sufficient. Nevertheless, this does not cause any problem to our externality argument, as we simply need to show that a collateral equilibrium cannot be constrained optimal, i.e. does not satisfy the necessary optimal condition (A.100).

[7]But under homothetic preferences the later is not possible. With a strictly concave utility function, the spot price varies with the market fundamental (is not constant), i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \neq 0$. As a result, when at least one of the collateral constraints is binding, i.e., $\mu^{\tilde{h}}_{cc-s} > 0$ for some type $\tilde{h}$, the last term in (A.100) will be non-zero. With this non-zero term, a collateral equilibrium will not be constrained efficient. It is true that, as an exceptional case, a collateral equilibrium could be a full first-best optimum, that is, the environment could be such that despite the focus of the paper we could ignore the collateral constraint.

Proof. We first prove that a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. The proof is based on the first-order conditions for Pareto program (A.98) and the first-order conditions for a competitive collateral equilibrium. Note that the resource constraints in the Pareto program (A.98) and the market-clearing constraints in the competitive collateral equilibrium are clearly equivalent. In addition, the collateral constraints are the same in both problems as well. Hence, we only need to match all first-order conditions from both problems. In addition, with limited space, we will focus only on the term that generates an externality.

Optimal Conditions for the Pareto Program (A.3)

Let μ^h_{cc-s} and $\mu^h_{\bar{u}}$ denote the Lagrange multipliers for the collateral constraint (A.95) for state s for an agent type h and for the participation constraint (A.99) for an agent type $h = 1, 2, \ldots, H$ with a normalization of $\mu^1_{\bar{u}} = 1$, respectively. Combining the first-order conditions with respect to c^h_{10} and k^h, and the complementarity slackness conditions for the collateral constraints gives:

$$\begin{aligned}\frac{u^h_{20}}{u^h_{10}} &= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s + \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} p'(z_s) \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \left[R_s k^h + \theta^{\tilde{h}}_{2s} \right] \\ &= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s}, \end{aligned} \tag{A.101}$$

where the last equation follows from the complementarity slackness condition with respect to collateral constraints:

$$\mu^{\tilde{h}}_{cc-s} \left\{ p(z_s) \left[R_s k^{\tilde{h}} + \theta^{\tilde{h}}_{2s} \right] + \theta^{\tilde{h}}_{1s} \right\} = 0 \Rightarrow \mu^{\tilde{h}}_{cc-s} \left[R_s k^{\tilde{h}} + \theta^{\tilde{h}}_{2s} \right] = - \frac{\mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s}}{p(z_s)}. \tag{A.102}$$

Note that (A.101) is exactly the same as (A.100).

Optimal Conditions for a Collateral Equilibrium

Let γ_{cc-s} be the Lagrange multiplier for the collateral constraint for state s. Combining the first-order conditions with respect to c^h_{10} and k^h gives:

$$\frac{u^h_{20}}{u^h_{10}} = \sum_s \pi_s \frac{\beta u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\gamma^h_{cc-s}}{u^h_{10}} p(z_s) R_s. \tag{A.103}$$

We are ready to prove the lemma.

(i) ($\Longleftarrow$) Suppose that $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. We then can show that any competitive collateral equilibrium allocation will also solve the Pareto program (A.98) by matching all necessary and sufficient conditions. In particular, we can pick $\frac{\mu_{20}}{\mu_{10}} = p_0$, $\frac{\mu_{\ell s}}{\mu_{10}} = Q_{\ell s}$, and $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}} = 0$. In conclusion, any collateral equilibrium allocation is constrained optimal if $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s.

(ii) ($\Longrightarrow$) Suppose that a competitive collateral equilibrium allocation is constrained optimal, i.e., solves the Pareto program (A.98). Hence, it must satisfy (A.101). Using the same matching conditions as above, this will be true only if the last terms in (A.101) is zero. We will prove this by a contradiction argument.

Suppose that there are some $\tilde{h}$ with $\mu^{\tilde{h}}_{cc-s} \neq 0$, and the last terms in (A.101) is zero:

$$\frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \sum_s \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} \right) = 0. \tag{A.104}$$

This must be true for all h and $\tilde{h}$.

We will now argue that $\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s}$ has the same *negative* sign for every state s. Using the first-order condition for the Pareto program with respect to θ^h_{1s}, we can show that

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} = \sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} - \beta \pi_s \sum_{\tilde{h}} \mu^{\tilde{h}}_{\bar{u}} u^{\tilde{h}}_{1s} \theta^{\tilde{h}}_{1s}, \tag{A.105}$$

where μ_{1s} is the Lagrange multiplier for the resource constraint for θ^h_{1s}. The resource constraint for θ^h_{1s}, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} = 0$, then implies that $\sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} = 0$ for all s. In addition, the first-order condition for the Pareto program with respect to c^h_{10} implies that $\mu^{\tilde{h}}_{\bar{u}} = \frac{\mu_{10} \alpha^{\tilde{h}}}{u^{\tilde{h}}_{10}}$. Thus, we now have

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} \right) \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s}. \tag{A.106}$$

The optimality requires that an agent with relative large IMRS, $\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}}$, will hold positive $\theta^{\tilde{h}}_{1s} \geq 0$ and vice versa. This implies that the positive term of $\alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} \geq 0$ will be weighted more than the negative one. Combining this result with the resource constraint for θ^h_{1s}, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} = 0$, we can conclude that $\sum_{\tilde{h}} \left(\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} \right) \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} \geq 0, \forall s$, and therefore

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} \right) \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} \leq 0, \forall s. \tag{A.107}$$

With strictly concave and identical homothetic utility function, we can show that $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} < 0$, and therefore can conclude that

$$\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} \right) \geq 0, \ \forall s. \tag{A.108}$$

As a result, (A.104) will hold only if

$$\sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} \right) \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} = 0, \forall s. \tag{A.109}$$

Given that $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta^{\tilde{h}}_{1s} = 0$, condition (A.109) implies that $\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{10}} = \frac{u^{h}_{1s}}{u^{h}_{10}}$, $\forall h, \tilde{h}; s$. Using the fact that $\frac{u^h_{2s}}{u^h_{1s}} = p(z_s)$ for all h, we can also show that $\frac{u^{\tilde{h}}_{1s}}{u^{\tilde{h}}_{1\hat{s}}} = \frac{u^{h}_{1s}}{u^{h}_{1\hat{s}}}$, $\forall h, \tilde{h}; s$. In words, the marginal rate of substitutions across times and states are equalized across agent types. Under the assumption stated in the proposition, these equalities are necessary and sufficient conditions for first-best optimality, which in turn implies that all collateral constraints are not binding, i.e., $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. Hence, we can conclude that a collateral equilibrium is constrained optimal, solving the Pareto program (A.98), only if all collateral constraints are not binding.

The rest of the proof is by contrapositive. Suppose a competitive collateral equilibrium is constrained optimal. The above result implies that a necessary and sufficient condition for a competitive collateral equilibrium to be constrained optimal is that all collateral constraints are not binding. No binding collateral constraints implies first-best optimality. In short, we have shown that first-best optimality is a necessary and sufficient condition for constrained optimality. Thus we can conclude that a competitive collateral equilibrium is constrained *suboptimal* if and only if it is not first-best optimal. □

In particular when the very last term in (A.100) is not zero so not first-best, we can show that it must be positive. As a result, the equilibrium price of good 2 in period $t = 0$ will be too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$. In addition, this implies that the competitive collateral equilibrium level of (endogenous) aggregate saving K^{ce} is too large[8] relative to the (constrained) optimal level of aggregate saving/collateral K^{op}. Intuitively, the planner can do better by lowering the aggregate saving/collateral. The result is summarized in the following proposition.

Proposition A.3. *With continuous, strictly concave, strictly increasing, and identical homothetic utility functions, if a competitive collateral equilibrium is not first-best optimal, then*

(i) the equilibrium price of good 2 in period $t = 0$, p_0, is too high, i.e., $p_0 > \left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$, and

[8]This is our analogous here to the result of Hart and Zingales (2013) that it is possible for agents to be saving too much.

(ii) the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$.

Proof. The proof is an immediate result of the proof of proposition A.2 above. First, if a competitive collateral equilibrium is not first-best optimal, then (by Proposition A.2) we can show that the last term of (A.101) is strictly positive:

$$\sum_{s} \frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} > 0. \tag{A.110}$$

This implies that the marginal rate of substitution between good 1 and good 2 in period $t = 0$ at the competitive collateral equilibrium is larger than the optimal level of the marginal rate of substitution between good 1 and good 2 in period $t = 0$, i.e., $\left.\frac{u_{20}^h}{u_{10}^h}\right|_{ce} > \left.\frac{u_{20}^h}{u_{10}^h}\right|_{op}$. This implies that the equilibrium price of good 2 in period $t = 0$ is too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u_{20}^h}{u_{10}^h}\right|_{op}$. In addition, given that the aggregate consumption of good 1 is fixed and preferences are identically homothetic, this result can be true only if the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$. □

This is also our simple version of Lorenzoni (2008) and the fire sales model. There is too much debt and hence too much collateral, and this is moving (distorting) prices.

F.4 Formal Definition of Equilibrium with Rights to Trade for the Collateral Economy

Let $\mathbf{x}^h(\mathbf{z}) = \left(\mathbf{c}_0^h, k^h, \delta^h(\mathbf{z}), \boldsymbol{\theta}^h(\mathbf{z}), \boldsymbol{\tau}^h(\mathbf{z}), \boldsymbol{\Delta}^h(\mathbf{z})\right)$ denote a typical bundle or allocation for an agent type h, where again $\boldsymbol{\Delta}^h(\mathbf{z}) \equiv \left[\Delta_s^h(z_s)\right]_s$. If $\delta^h(\mathbf{z}) = 0$, then the rest of the $\mathbf{z}$ contingent choices need not be specified, as agent h is choosing not to trade at $\mathbf{z}$.

Definition A.8. *A competitive equilibrium with segregated exchanges indexed by z is a specification of allocation $\left[\mathbf{x}^h(\mathbf{z})\right]_{h,\mathbf{z}}$ and prices $(p_0, \mathbf{Q}, \mathbf{P}_\Delta, \mathbf{p})$ such that*

(i) for any agent type h as a price taker, allocation $\left[x^h(\mathbf{z})\right]_{\mathbf{z}}$ solves

$$\max_{\left[\mathbf{x}^h(\mathbf{z})\right]_{\mathbf{z}}} \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[u\left(c_{10}^h, c_{20}^h\right) + \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s}^h(z_s) + \tau_{1s}^h(z_s), e_{2s}^h + R_s k^h + \theta_{2s}^h(z_s) + \tau_{2s}^h(z_s)\right)\right]$$

subject to collateral constraints

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[p(z_s)\left[R_s k^h + \theta_{2s}^h(z_s)\right] + \theta_{1s}^h(z_s)\right] \geq 0, \forall s, \tag{A.111}$$

budget constraint at $t = 0$

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ c_{10}^h + p_0 \left[c_{20}^h + k^h \right] + \sum_s \sum_\ell Q_\ell(z_s, s)\, \theta_{\ell s}^h(z_s) + \sum_s P_\Delta(z_s, s)\, \Delta_s^h(z_s) \Big\} \leq e_{10}^h + p_0 e_{20}^h, \tag{A.112}$$

and budget constraint at state s

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[\tau_{1s}^h(z_s) + p(z_s)\, \tau_{2s}^h(z_s) \right] = 0, \forall s; \tag{A.113}$$

(ii) markets clear in good 1 in period $t = 0$

$$\sum_h \alpha^h c_{10}^h = \sum_h \alpha^h e_{10}^h, \tag{A.114}$$

markets clear in good 2 in period $t = 0$

$$\sum_h \alpha^h \left[c_{20}^h + k^h \right] = \sum_h \alpha^h e_{20}^h, \tag{A.115}$$

markets clear in securities

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z})\, \alpha^h \theta_{\ell s}^h(z_s) = 0, \forall s, \ell, z_s, \tag{A.116}$$

markets clear in spot trades

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z})\, \alpha^h \tau_{\ell s}^h(z_s) = 0, \forall s, \ell, z_s, \tag{A.117}$$

and markets clear in rights to trade

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z})\, \alpha^h \Delta_s^h(z_s) = 0, \forall s, z_s, \tag{A.118}$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states except z_s in state s.

We highlight that from the market clearing condition of the rights to trade (A.118), for any active spot market chosen by multiple types, spot markets must clear and in this sense the valuation of collateral p_s chosen in an active market is self-fulfilling, a fixed point.

F.5 Numerical Examples for the Collateral Economy

F.5.1 The Collateral Economy without Uncertainty: Intertemporal Smoothing

This example is in principle equivalent to the leading example the main text. Here we allow for borrowing and lending. Regardless, there will be no borrowing and lending in equilibrium for this particular example. That is why we can shut down the borrowing and lending channel in the main text with no loss of generality.

There are two periods, $t = 0, 1$, and a single state, $S = 1$ in period $t = 1$. So this is a pure intertemporal economy. We thus make the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades on financial options. Indeed, in this example economy no securities will be traded, in equilibrium, and in this way we focus on the market for rights to trade in spot markets, only. Henceforth we drop all subscript s from the notation.

There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function[9]

$$u(c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}. \tag{A.119}$$

Each type h consists of $\frac{1}{2}$ fraction of the population, i.e., $\alpha^h = \frac{1}{2}$. In addition, the common discount factor is $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table A.5 below. Note that an agent type 1 is well endowed with both goods in period $t = 0$ relative to $t = 1$, and vice versa for type 2. The first best allocation has both agents consuming 2 units of each good in every period. The first-best allocation with full commitment and hence no collateral constraints thus suggests the obvious, that agent type 2 would like to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow, and therefore will be constrained in the competitive collateral equilibrium. Borrowing requires collateral, and agent type 2 is short of this as well. The equilibrium with the externality present, and also the one with the externality corrected, will have agent type 2 borrowing nothing and only trading in spot markets. Agent type 1 will be saving on its own to smooth consumption over time.

We summarize the equilibrium allocation in Table 1 featuring collateral k^h and consumption $c^h_{\ell s}$. The derivation of the competitive equilibrium with the externality is presented below.

There is no loss of generality to consider a solution with no security trading, i.e., $\theta^h_{\ell s} = 0$ for all

[9] Here we set the coefficient of relative risk aversion $\gamma = 2$, and drop the irrelevant constant term.

Table A.5: Endowment profiles of the agents.

	endowments				equilibrium with the externality (ex)					equilibrium with rights to trade (op)				
	e_{10}^h	e_{20}^h	e_{11}^h	e_{21}^h	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h	k^h	c_{10}^h	c_{20}^h	c_{11}^h	c_{21}^h
$h=1$	3	3	1	1	1.360	2.690	1.776	1.325	1.776	1.175	2.607	1.841	1.297	1.678
$h=2$	1	1	3	3	0	1.310	0.865	2.675	3.584	0	1.393	0.984	2.703	3.497

h and for all ℓ. Agents do however actively trade in spot markets. With the externality (denoted "ex"), the price of good 2 in period $t=0$ is $p_0^{ex} = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and the market fundamental in period $t=1$ is $z^{ex} = \frac{4}{4+k^{ex}} = 0.7463$, which implies that the spot price is $p\left(z^{ex}\right) = 0.5570$.

We will now turn to a corresponding competitive equilibrium with rights to trade in segregated exchanges (without the externality, the one with rights to trade that we have featured). There is *one and only one active spot market at* $t=1$, $z^{op} = 0.7729$ ("op" stands for optimality; we have not yet proved the first welfare theorem for markets with rights to trade, but it will apply), even though all spot markets are available in principle for trade. That is, in equilibrium, both types optimally choose to trade in the same spot market with specified market fundamental, $z^{op} = 0.7729$. Note that as anticipated there is less saving with the externality corrected, so the spot price of good 2 is lower at $t=0$ and is higher at $t=1$ relative to the equilibrium with externalities present.

Table A.6 presents equilibrium prices/fees of rights to trade in spot markets, that is $P_\Delta(z)$ not only for z^{op} but also other, different market fundamental levels z. We are here defining $P_\Delta(z) = P_\Delta(z_s, s)$ where there are no states so we have dropped the s notation. Note again that the prices/fees of non-active spot markets are available, but at such prices agents do not want to trade in them.

Table A.6: Equilibrium prices of rights to trade in spot markets $P_\Delta(z)$. Bold numbers are equilibrium prices for actively traded spot markets.

	$z=0.7479$	$\mathbf{z=0.7729}$	$z=0.7979$
$P_\Delta(z)$	0.4639	**0.5375**	0.6118

An agent type 1 is coming in with good 2 in storage, and therefore his right is positive. Type 1 pays for right to trade. This makes sense as agent type 1 is doing the saving in good 2 and there is too much saving in the (ex) equilibrium. On the other hand, an agent type 2's right is negative. Thus, with a positive equilibrium fee $P_\Delta(z^{op}) = 0.5375$, an agent type 2 will be paid for her willingness to choose that market. Agent type 2 is facing a higher price of the good 2, that will be purchased. But there is compensation. In particular, a constrained agent ($h = 2$) with $\Delta^2(z^{op}) = -0.6813$, is receiving $-P_\Delta(z^{op})\Delta^2(z^{op}) = 0.3662$ in period $t = 0$ for being in the spot market $z^{op} = 0.7729$. Graphically, this shifts her budget line outward at $t = 0$ by $T = 0.3662$, hence in the direction of being less constrained.[10]

Derivation of a Competitive Equilibrium with Externality for Example F.5.1

The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We now solve for an equilibrium k. From the market clearing conditions of contracts, we can set $\theta^1_{11} = -\theta^2_{11} = \hat{\theta}$ and $\theta^1_{21} = -\theta^2_{21} = \theta$. Note that this does not mean agent 1 is demanding both securities. In addition, using the specified collateral allocation, the market fundamental in period $t = 1$ is now $z = \frac{4}{4+k}$ (the ratio of endowment of good 1 to the sum of endowment of good 2 and saving), and consequently the spot price of good 2 in period 1 is $p(z) = \left(\frac{4}{4+k}\right)^2$.

With homothetic preferences, the first-order conditions of the problem (A.86) for both types imply that in spot markets at date $t = 0$

$$p_0 = \left(\frac{c^1_{10}}{c^1_{20}}\right)^2 = \left(\frac{c^2_{10}}{c^2_{20}}\right)^2 = \left(\frac{4}{4-k}\right)^2. \tag{A.120}$$

Since agent 1's collateral constraint is not binding, the first-order conditions of her utility-maximization problem (A.86) with respect to θ^1_{21} and c^1_{10} lead to

$$Q_{21} = \frac{u^1_{21}}{u^1_{10}} = \left(\frac{c^1_{10}}{c^1_{21}}\right)^2, \tag{A.121}$$

where $u^h_{it} = \frac{\partial u^h}{\partial c_{it}}$ is the marginal utility with respect to c_{it}, and Q_{21} is the price of a security paying in good 2 in period $t = 1$. Note that we put superscript h on the utility function for clarity.

[10] Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium. The expected utility of an agent type 1 and type 2 in this competitive equilibrium with segregated exchanges (without the externality) are $U^1_{op} = -2.2936, U^2_{op} = -2.3905$, respectively. The expected utility of an agent type 1 and type 2 in the competitive collateral equilibrium allocation (with the externality) are $U^1_{ex} = -2.2527$ and $U^2_{ex} = -2.5724$, respectively. Thus if nothing else were done, internalizing the externality would be beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we also prove below.

Further, the first-order conditions of the consumer's problem (A.86) with respect to θ^1_{21} and k^1 (interior solutions) lead to

$$p_0 = Q_{21}. \tag{A.122}$$

Intuitively, this is the case because their payoffs are identical and both are collateralizable. Using (A.120) and (A.121), condition (A.122) implies that

$$\frac{c^1_{10}}{c^1_{20}} = \frac{c^1_{10}}{c^1_{21}} \Longrightarrow c^1_{20} = c^1_{21}. \tag{A.123}$$

That is, an unconstrained agent consumes the same amount of good 2 in both periods.

Substituting (A.120) and (A.121) into (A.122) gives

$$\begin{aligned} \left(\frac{4}{4-k}\right)^2 &= \left(\frac{c^1_{10}}{c^1_{21}}\right)^2; \\ \frac{4}{4-k} &= \frac{c^1_{10}}{1+k+\theta} \Longrightarrow (4-k)\,c^1_{10} = 4+4k+4\theta, \end{aligned} \tag{A.124}$$

where we use $c^1_{21} = 1+k+\theta$.

On the other hand, an agent type 2's collateral constraint is binding; with $k^2 = 0$,

$$\hat{\theta}^2 + p(z)\theta^2 = 0 \Longrightarrow -\hat{\theta} - p(z)\theta = 0 \Longrightarrow \hat{\theta} = -\left(\frac{4}{4+k}\right)^2 \theta, \tag{A.125}$$

where the second and the last equations use $\hat{\theta}^2 = -\hat{\theta}$ and $\theta^2 = -\theta$, and $p(z) = \left(\frac{4}{4+k}\right)^2$, respectively.

The budget constraint of an agent 1 (A.88) can be written as

$$c^1_{10} - 3 + p_0\left[c^1_{20} + k - 3\right] + Q_{11}\hat{\theta} + Q_{21}\theta \;\; = \;\; 0. \tag{A.126}$$

A standard no-arbitrage argument (similar to the one used in Lemma A.7) implies that

$$Q_{21} = p(z)Q_{11}. \tag{A.127}$$

It thus true from (A.127) that

$$Q_{11}\hat{\theta} + Q_{21}\theta = Q_{11}\hat{\theta} + Q_{11}p(z)\theta = Q_{11}\left[\hat{\theta} + p(z)\theta\right]p(z) = 0, \tag{A.128}$$

where the last equation follows the fact that the term in the bracket is zero, from (A.125). Now the LHS of the budget constraint (A.126) can be rewritten as

$$c^1_{10} + p_0\left[c^1_{20} + k - 3\right] = 3. \tag{A.129}$$

Using (A.120), we can replace c^1_{20} by $\left(\frac{4-k}{4}\right) c^1_{10}$. Then using $p_0 = \left(\frac{4}{4-k}\right)^2$ gives

$$\begin{aligned} c^1_{10} + \left(\frac{4}{4-k}\right)^2 \left[\left(\frac{4-k}{4}\right) c^1_{10} + k - 3\right] &= 3 \\ \Longrightarrow (4-k)\, c^1_{10} &= \frac{3k^2 - 40k + 96}{8-k}. \end{aligned} \tag{A.130}$$

Substituting (A.124)into (A.130) gives

$$\frac{3k^2 - 40k + 96}{8-k} = 4 + 4\theta + 4k \Longrightarrow 4\theta + 4k = \frac{3k^2 - 36k + 64}{8-k}. \tag{A.131}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z = \frac{4}{4+k} = \frac{c^1_{11}}{c^1_{21}} \Longrightarrow \frac{4}{4+k} = \frac{1+\hat{\theta}}{1+k+\theta}. \tag{A.132}$$

Substitute (A.125) into (A.132) gives

$$4\theta + 4k = -3k\left(\frac{4+k}{8+k}\right) + 4k. \tag{A.133}$$

Using (A.131) and (A.133), we have

$$\frac{3k^2 - 36k + 64}{8-k} = -3k\left(\frac{4+k}{8+k}\right) + 4k \Longrightarrow 4k^3 - 384k + 512 = 0. \tag{A.134}$$

There are three roots for equation (A.134). Using the condition that $0 \le k \le 4$, there is only one feasible solution, i.e. $k \approx 1.3595$. To sum up, the equilibrium collateral allocation is $k^1 = k = 1.3595$ and $k^2 = 0$.

F.5.2 The Collateral Economy with State Contingent Securities

This example illustrates an economy with uncertainty where collateralized securities, θ^h_{1s}, are actively traded. All agents are constrained, but at different states. In particular, an agent will be binding in a state where her endowment is large, as it is for such states that she would make a promise to pay, and promises must be backed by collateral.

The economy in this example is similar to the one in Environment F.5.1 with two periods, but there are two states, $S = 2$ at $t = 1$. The endowment profiles are presented in Table A.7. Note, unlike the first example, that the agents are identical in endowments at $t = 0$. But agent type 1 has relatively more of both goods in state $s = 1$ than in state $s = 2$, and vice versa for agent type 2.

Table A.7: Endowment profiles of the agents.

	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

We will now solve for a competitive equilibrium with the externality. The detailed derivation is presented below. An agent type $h = 1$ issues $\theta^1_{11} = -0.3042$ units of collateralized security paying good 1 at $s = 1$, that is, promises to pay at $s = 1$, and invests the same amount $\theta^1_{12} = 0.3042$, to be paid at $s = 2$. Vice versa for an agent $h = 2$. We now turn to the competitive equilibrium with rights to trade. Each type of agent holds the same amount of collateral good $0.4200 = k^{op} < k^{ex} = 0.4603$, less than the one in competitive equilibrium with the externality, as anticipated. Collateralized securities are of the same sign but overall payments are less $\theta^1_{11} = -\theta^1_{12} = -0.2872 = -\theta^2_{11} = \theta^2_{12}$. That is, there is less volume in the securities markets relative to the equilibrium with the externality. This is again because the agents save less, consistent with issuing fewer securities, hence less collateral.

Derivation of a Competitive Equilibrium with Externality for Example F.5.2

First of all, the symmetry of the environment implies that the equilibrium collateral allocation is also symmetric, i.e. $k^h = k$ for all h. As a result, the price of good 2 in period $t = 0$ is given by

$$p_0 = \left(\frac{2}{2-k}\right)^2, \tag{A.135}$$

and the spot price of good 2 in each state s is given by

$$p_s = \left(\frac{2}{2+k}\right)^2, \forall s. \tag{A.136}$$

Further, the price of a (collateralized) security paying in good 2 in state s is given by

$$Q_{2s} = \max_h \left(\frac{\pi_s u^h_{2s}}{u^h_{10}}\right), \ \forall s. \tag{A.137}$$

The endowment structure implies that agents type 2 will have higher MRS $\frac{\pi_s u^h_{2s}}{u^h_{10}}$ in state 1, and vice versa. In addition, the structure also implies that $\theta^1_{21} = \theta^2_{22} = \theta$ and $\theta^1_{11} = \theta^2_{12} = \hat{\theta}$. Hence, (A.137) can be rewritten as

$$Q_{21} = \frac{\pi_s u^2_{21}}{u^2_{10}} = \frac{1}{2}\left(\frac{2}{1+k^1+\theta^1_{21}}\right)^2 = \frac{1}{2}\left(\frac{2}{1+k+\theta}\right)^2 = \frac{1}{2}\left(\frac{2}{1+k^2+\theta^2_{22}}\right)^2 = \frac{\pi_s u^1_{22}}{u^1_{10}} = Q_{22}. \tag{A.138}$$

That is, the symmetry structure implies that $Q_{21} = Q_{22}$. Using the optimal conditions with respect to k^h and θ^h_{2s}, we can show that

$$p_0 = Q_{21} + Q_{22} \Longrightarrow \left(\frac{2}{2-k}\right)^2 = \left(\frac{2}{1+k+\theta}\right)^2. \tag{A.139}$$

Next, with the homotheticity of preferences, the ratio of consumption in each state of each agent must be equal to the market fundamental; that is,

$$\frac{1+\hat{\theta}}{1+k+\theta} = \frac{2}{2+k}. \tag{A.140}$$

Furthermore, the collateral constraint in state $s=1$ of an agent type $h=1$ is binding, i.e.

$$p_1 k - \hat{\theta} - p_1\theta = 0 \Longrightarrow \hat{\theta} = \left(\frac{2}{2+k}\right)^2 (k-\theta). \tag{A.141}$$

Note that the same equation can be derived from the binding collateral constraint in state $s=2$ for an agent type $h=2$.

We can compute a collateral equilibrium using (A.139), (A.140), and (A.141) to solve for $\left(k, \theta, \hat{\theta}\right)$. We can rewrite (A.139) as

$$2-k = 1+k+\theta \Longrightarrow \theta = 1-2k. \tag{A.142}$$

In addition, Substituting (A.141) into (A.140) gives

$$1 + \left(\frac{2}{2+k}\right)^2 (k-\theta) = \left(\frac{2}{2+k}\right)(1+k+\theta). \tag{A.143}$$

Then, substituting (A.142) into (A.143) will give

$$\begin{aligned} 1 + \left(\frac{2}{2+k}\right)^2 (k-1+2k) &= \left(\frac{2}{2+k}\right)(1+k+1-2k) \\ &\Longrightarrow 3k^2 + 16k - 8 = 0. \end{aligned} \tag{A.144}$$

The unique feasible (positive) solution to the above quadratic equation is $k \approx 0.4603$.

F.5.3 The Collateral Economy with Heterogeneous Borrowers and The Role of Mixtures/Lottery

This example breaks new ground and presents an economy where it is possible to assign agents to different exchanges and to have multiple segregated exchanges. We return to the collateral economy for simplicity of notation and ideas.

Apart from more heterogeneity in agent types, all other aspects of the environment are as in Environment F.5.1- where agent type 1 was lender/saver. But now there are three types of agents, with two borrower types 2, 3. Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$. The endowment profiles are given in Table A.8, below. As in Environment F.5.1, there is no uncertainty.

Table A.8: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	4.26	11.5	0.5	0.5
$h = 2$	3.92	0.5	7	5
$h = 3$	4.32	0.5	5	7

Yet interestingly, there are now *two active spot markets*, $z = 0.6113$ and $z = 0.8132$ in the competitive equilibrium with segregated exchanges.[11] The spot market $z = 0.6113$ consists of some fraction of agents type 1 (19.69 percent), and all of agents type 3 (that is, all of the constrained agents of a certain type). On the other hand, the spot market $z = 0.8132$ consists of some residual fraction of agents type 1 (80.31 percent) and all of agents type 2 (all of the other constrained type). We use the term mixtures to refer to the fact that agent 1 is allocated to two active markets in some nontrivial proportions. See Table A.9 below for the equilibrium allocation.

We can now see how the market-clearing constraints for lotteries (A.15) work and how the intermediary work in this example. For notational purposes, let $w(\cdot)$ denote a particular bundle as specified in the table. For the first bundle $w(1)$, the market clearing condition implies that the intermediary issues $0.1969 \times \frac{1}{3} = 0.0656$ units of bundle $w(1)$. Similarly, it issues 0.2677 units of bundle $w(2)$, 0.3333 units of bundle $w(3)$, and 0.3333 units of bundle $w(4)$. The key point here is that the intermediary is the supplier of the bundles, which are demanded by the agents.

Equilibrium fees of rights to trade in spot markets, including the fees of inactive spot markets are summarized in Table A.10 below.[12]

It is socially optimal to compensate constrained agents with positive transfers at period $t = 0$, to try to move back toward the first best, i.e., alleviate borrowing constraints. In this example the

[11]Note that the active markets are discretely separated from one another, i.e., the separation is not numerical but real. We have computed equilibrium for various refinements of the underlying space of zs and one gets discrete separation, as in Table A.10, where there are two inactive exchanges between the active zs. We conjecture there are as many active markets as there are constrained types but have been unable to prove.

[12]Of course, ex-post we could have shut the inactive ones down, but we could not know which a priori.

Table A.9: Equilibrium allocation of (non-zero-mass) lotteries. There are multiple active security exchanges; $z = 0.6113$ and $z = 0.8132$.

	$h = 1$		$h = 2$	$h = 3$
	$w(1)$	$w(2)$	$w(3)$	$w(4)$
k	6.3082	4.6072	0.0000	0.0000
$\hat{\tau}$	1.3892	1.6384	-1.3159	-0.2735
τ	-3.7176	-2.4776	1.9898	0.7319
c_{10}	5.6204	5.6204	4.4835	2.3961
c_{20}	3.3982	3.3982	2.7106	1.4491
c_{11}	1.8892	2.1384	5.6841	4.7265
c_{21}	3.0905	2.6296	6.9898	7.7319
z	**0.6113**	**0.8132**	**0.8132**	**0.6113**
Δ	3.6618	3.6532	-2.9340	-0.7209
x^h	0.1969	0.8031	1.0000	1.0000
U^h	-1.3211		-0.9110	-1.4483

Table A.10: Equilibrium fees of rights to trade in spot markets. The bold numbers are the equilibrium prices of actively traded exchanges.

	$z = 0.6088$	$\mathbf{z = 0.6113}$	$z = 0.6138$	$z = 0.8088$	$\mathbf{z = 0.8132}$	$z = 0.8138$
$P_\Delta(z)$	0.9119	**0.9348**	0.9589	2.2339	**2.2537**	2.2564

number of active segregated spot markets is equal to the number of constrained types, to allow this to happen.

Specifically, in the competitive equilibrium with segregated exchanges (without the externality), the right from the market fundamental of both constrained types are negative, i.e., $\Delta^2 = -2.9340$ and $\Delta^3 = -0.7209$. With a positive equilibrium price of the right in each active market, agents type 2 and agents type 3 are paid for the chosen rights to trade at fees $P_\Delta(z)\,\Delta^h(z)$. Agents type 1 buy a lottery which is actuarially fair; fees are paid in proportion to the relative number of its type assigned to each exchange. Agents type 1 would like to buy into the higher spot market, $z = 0.8132$ in this case, with certainty, where good 2 is more valuable because with her endogenous saving she would benefit, but such a deterministic choice is not affordable.

From the perspective of type 1, its purchased assignment is a lottery and its assignment into one or the other of the two active segregated exchanges is random. From the perspective of the economy-wide equilibrium a fixed fraction of type 1 are assigned to one or the other of the two active segregated exchanges, with fractions adding to one, and fractions equal to the probabilities from the perspective of the individual agent. We shall need a broker-dealer as an intermediary to allow this pooling to happen, and then we can define formally the necessary extension of the definition of competitive equilibrium with rights to trade in segregated exchanges.

Finally, note that the example is of course generating consumption allocations $\mathbf{c}$, saving k, and transfer $\boldsymbol{\tau}$ as excess demands. We could have mentioned security trades $\boldsymbol{\theta}$ as in Example F.5.2, but here as in Example F.5.1 there are no active security trades, so these were all zero. The right Δ in state s is simply a scaled version of excess demand in state s, as was made clear earlier. Likewise, the example is in terms of fundamental z but we could as easily rewritten the problem in terms of spot price $p(z)$. We continue to use Δ and z in the remainder of this section, but the reader should note yet again in the collateral environment there is an entirely equivalent formulation in terms of p; the only difference is notation.

Derivation of a Competitive Equilibrium with Externality for Example F.5.3

We restrict our attention to a symmetric allocation of each type. Using Lemma A.4, we assume that all constrained agents hold no collateral, i.e., $k^h = 0$ for $h = 2, 3$. Let $k^1 = k$.

First, the first-order conditions of the consumer's problem (A.86) result in

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^2}{c_{20}^2} = \frac{c_{10}^3}{c_{20}^3} = \frac{12.5}{12.5 - k}. \tag{A.145}$$

From the endowment profile, it is clear that an agent 1 will not be constrained. The first-order conditions of the consumer's problem (A.86) with respect to θ^1_{21} and c^1_{10} lead to

$$\frac{u^1_{21}}{u^1_{10}} = Q_{21}. \tag{A.146}$$

Further, the first-order conditions of the consumer's problem (A.86) with respect to θ^1_{21} and k^1 (interior solutions) lead to

$$p_0 = Q_{21}. \tag{A.147}$$

Combining (A.146), (A.147) and the utility function (A.119), gives

$$p_0 = \left(\frac{12.5}{12.5-k}\right)^2 \quad = \quad Q_{21} = \frac{u^1_{21}}{u^1_{10}} = \left(\frac{c^1_{10}}{c^1_{21}}\right)^2. \tag{A.148}$$

This implies that

$$\frac{12.5}{12.5-k} \quad = \quad \frac{c^1_{10}}{c^1_{21}} = \frac{c^1_{10}}{0.5+k+\theta^1_{21}} \Longrightarrow (12.5-k)\, c^1_{10} = 12.5\left(0.5+k+\theta^1_{21}\right), \tag{A.149}$$

where we use $c^1_{21} = 0.5 + k + \theta^1_{21}$.

In addition, the market fundamental in period $t = 1$ is $z = \frac{12.5}{12.5+k}$, and consequently the spot price of good 2 in period $t = 1$ is $\left(\frac{12.5}{12.5+k}\right)^2$. The bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities, imply that

$$\theta^1_{11} = -\left(\frac{12.5}{12.5+k}\right)^2 \theta^1_{21}. \tag{A.150}$$

A standard no-arbitrage argument (similar to the one used in Lemma A.7) implies that

$$Q_{21} = p(z)Q_{11}, \tag{A.151}$$

which can be used to show that

$$Q_{11}\theta^1_{11} + Q_{21}\theta^1_{21} = Q_{11}\theta^1_{11} + Q_{11}p(z)\theta^1_{21} = Q_{11}\left[\theta^1_{11} + p(z)\theta^1_{21}\right]p(z) = 0, \tag{A.152}$$

where the last equation follows the bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities. The budget constraint of an agent 1 (A.88) can be written as

$$c^1_{10} - e^1_{10} + p_0\left[c^1_{20} + k - e^1_{20}\right] \quad = \quad 0. \tag{A.153}$$

Substituting (A.145) and (A.148) into (A.153), we have

$$(12.5-k)\, c^1_{10} \quad = \quad \frac{12.5^2\left(e^1_{20} - k\right) + e^1_{10}\,(12.5-k)^2}{25-k}. \tag{A.154}$$

Substituting (A.149) into (A.154), we have

$$12.5\left(0.5+k+\theta_{21}^{1}\right) \quad = \quad \frac{12.5^{2}\left(e_{20}^{1}-k\right)+e_{10}^{1}\left(12.5-k\right)^{2}}{25-k}. \tag{A.155}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z=\frac{12.5}{12.5+k}=\frac{c_{11}^{1}}{c_{21}^{1}} \Longrightarrow \frac{12.5}{12.5+k}=\frac{0.5+\theta_{11}^{1}}{0.5+k+\theta_{21}^{1}}, \tag{A.156}$$

where the equality follows (A.150). This can be rewritten as

$$12.5\left(0.5+k+\theta_{21}^{1}\right)=(12.5+k)\left(0.5-\left(\frac{12.5}{12.5+k}\right)^{2}\theta_{21}^{1}\right). \tag{A.157}$$

Solving (A.155) and (A.157) for k and θ_{21}^{1}, with $e_{10}^{1} = 4.2631$ and $e_{20}^{1} = 11.5$, gives one feasible solution $(0 \leq k \leq 12.5)$ $k = 7.2836, \theta_{21}^{1} = -4.2849$. To sum up, the competitive collateral equilibrium allocation is $k^1 = k = 7.2836$, and $k^2 = k^3 = 0$.

F.6 More Results

Lemma A.2. *For any state-contingent security, there exists a security with* no default *that can generate the* same total payoffs *using the* same amount of collateral*.*

Proof of Lemma A.2: Default is Irrelevant under Complete Contracts. Consider a contingent security *that will be in default* in state s, with collateral $\widehat{C} < \frac{1}{R_s p(z_s)}$. That is, an issuer of this security will "default" in state s. Hence, the payoff of this security (in units of good 1) in state s is

$$\min\left(1, \widehat{C}R_s p(z_s)\right) = \widehat{C}R_s p(z_s) < 1. \tag{A.158}$$

We now argue that there is an alternative security that does not default but generates exactly the same total payoffs using the same amount of collateral overall. Consider a state-s contingent security with collateral amount $\widetilde{C} = \frac{1}{R_s p(z_s)}$. This security will not default. It is straightforward to show that the payoff of this security is one unit of good 1 in state s. Now consider $\widehat{C}R_s p(z_s)$ units of the *alternative* security. That collection of securities pays in state s one per unit or $\widehat{C}R_s p(z_s)$ in total. This is exactly the same as the payoff of the original security with default: see (A.158). In addition, the total collateral for $\widehat{C}R_s p(z_s)$ units of the alternative security with $\frac{1}{R_s p(z_s)}$ collateral per unit is $\widehat{C}$, which is exactly the same as the collateral level of the original security. Therefore, the alternative security can generate the same payoffs using the same total amount of collateral but without default. A similar argument also applies to all other types of securities. □

F.6.1 Details of the Building Blocks of the Collateral Constraints

This section precisely defines directly collateralized and asset-back securities (pyramiding), and derives the unified collateral constraints (A.87) by considering the collateral constraints of each type of securities one at a time and adding them up (and disaggregating back down).

Collateral Constraints on Directly Collateralized Securities

Let ψ_{1s}^h and ψ_{2s}^h denote agent h's demand at the end of period 0 for a security paying in good 1 and in good 2, both with good 2 as collateral directly, respectively. Again, we adopt the convention that positive means demand and negative means sale. So, holding a positive amount of a security paying good 2 in state s, $max\left(0, \psi_{2s}^h\right) = \psi_{2s}^h$, a positive number, is equivalent to buying that security (or lending) while holding a negative amount of a security, $min\left(0, \psi_{2s}^h\right) = \psi_{2s}^h$, a negative number, is equivalent to selling that security (or borrowing). In short, the max and min operators pick off demand and supply, respectively. A wedge is created by the need to back the supply by collateral but not the demand.

More generally, a security paying a unit of good 1 in state s backed by good 2 pays the minimum of 1 unit of good 1 or the value of its collateral in state s. By an argument similar to the one given earlier, the minimum no-default collateral is $\frac{1}{p(z_s)R_s}$ per unit. Similarly, with no-default and no-over-collateralization, a security paying in good 2 in state s requires $\frac{1}{R_s}$ units of good 2 as collateral. The results so far are summarized in the first two rows of the Table A.11 with collateral requirement in the last column.

For securities $\left(\psi_{1s}^h, \psi_{2s}^h\right)$ with good 2 as collateral, paying in good 1 and good 2, respectively, agent h must hold good 2 at the end of period 0 no less than the collateral requirement in any state (shown in Table A.11):

$$k^h \geq -\min\left(0, \psi_{1s}^h\right)\left(\frac{1}{R_s p(z_s)}\right) - \min\left(0, \psi_{2s}^h\right)\left(\frac{1}{R_s}\right), \ \forall s, \tag{A.159}$$

which can be rewritten as

$$p(z_s)R_s k^h + \min\left(0, \psi_{1s}^h\right) + p(z_s)\min\left(0, \psi_{2s}^h\right) \geq 0, \ \forall s. \tag{A.160}$$

These are *state-contingent collateral requirement constraints* with directly collateralized securities. We incorporate asset-backed securities in the next section.

Note that when an agent h's collateral requirement constraints (A.159) are not binding for every state s (i.e., the LHS of (A.159) exceeds its RHS or (A.159) holds with strict inequality for every

Table A.11: Collateral requirements for each type of securities.

	payment unit	collateral unit	issued liabilities	purchased assets available as collateral	total collateral requirement for no default securities
ψ^h_{1s}	good 1	good 2	$-\min\left(0, \psi^h_{1s}\right)$	$\max\left(0, \psi^h_{1s}\right)$	$-\left(\frac{1}{R_s p(z_s)}\right)\min\left(0, \psi^h_{1s}\right)$
ψ^h_{2s}	good 2	good 2	$-\min\left(0, \psi^h_{2s}\right)$	$\max\left(0, \psi^h_{2s}\right)$	$-\left(\frac{1}{R_s}\right)\min\left(0, \psi^h_{2s}\right)$
σ^h_{1s}	good 1	securities paying in good 2	$-\min\left(0, \sigma^h_{1s}\right)$	$\max\left(0, \sigma^h_{1s}\right)$	$-\left(\frac{1}{p(z_s)}\right)\min\left(0, \sigma^h_{1s}\right)$
σ^h_{2s}	good 2	securities paying in good 1	$-\min\left(0, \sigma^h_{2s}\right)$	$\max\left(0, \sigma^h_{2s}\right)$	$-p(z_s)\min\left(0, \sigma^h_{2s}\right)$
ν^h_{1s}	good 1	securities paying in good 1	$-\min\left(0, \nu^h_{1s}\right)$	$\max\left(0, \nu^h_{1s}\right)$	$-\min\left(0, \nu^h_{1s}\right)$
ν^h_{2s}	good 2	securities paying in good 2	$-\min\left(0, \nu^h_{2s}\right)$	$\max\left(0, \nu^h_{2s}\right)$	$-\min\left(0, \nu^h_{2s}\right)$

state s), then the agent h holds collateral k^h more than needed to back issued securities. The extra part of collateral is normal saving.

Pyramiding: Asset-Backed Securities

In real world economies, agents are allowed to use the *promises to receive* goods of others as collateral to back their own promises. This is termed *pyramiding*. In other words, there are two types of collateral, good 2 itself (described in the preceding section) and "assets" backed by such collateral. The prototypical example of an asset-backed promise in this paper is an ex-ante agreement for an agent to give up good 1 in the spot market in state s backed by someone else's promise, a receipt of good 2, or vice versa. The promise of receipt is the asset, and this backs the promise to pay. Indeed, if the planned spot-market trade is at equilibrium price of $p(z_s)$, then one is moving along a budget line and so the value of collateral, the good to be recovered, exactly equals the promise and there is no need for additional underlying collateral.

With two physical commodities, there are four possible types of asset-backed securities, summarized in the last four rows of Table A.11. For example, a unit of an asset-backed security $\hat{\sigma}_s$ paying in good 1 in state s needs $\frac{1}{p(z_s)}$ units of assets paying in good 2 as collateral. The value of the payoff of $\frac{1}{p(z_s)}$ units of securities paying in good 2 in state s equals $p(z_s) \times \frac{1}{p(z_s)} = 1$ unit of good 1, which is exactly the face-value promise to pay. These collateral requirements are minimum no-default levels.

As shown in the third row of Table A.11 (see the column titled total collateral requirement), an asset-backed security paying a unit of good 1 in state s, σ^h_{1s}, requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\left(\frac{1}{p(z_s)}\right)\min\left(0,\sigma^h_{1s}\right)$. Similarly, an asset-backed security ν^h_{2s} requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\min\left(0,\nu^h_{2s}\right)$ (see the last row of Table A.11). On the other hand, the total amount of purchased assets paying in good 2 is $\max\left(0,\psi^h_{2s}\right)+\max\left(0,\sigma^h_{2s}\right)+\max\left(0,\nu^h_{2s}\right)$, as shown in the second, fourth and last rows of Table A.11 (see the next-to-last column titled purchased assets). Hence, the collateral requirement condition regarding issued securities σ^h_{1s} and ν^h_{2s} that require financial assets paying in good 2 as collateral can be written as, for any state s,

$$\max\left(0,\psi^h_{2s}\right)+\max\left(0,\sigma^h_{2s}\right)+\max\left(0,\nu^h_{2s}\right) \quad \geq \quad -\left(\frac{1}{p(z_s)}\right)\min\left(0,\sigma^h_{1s}\right)-\min\left(0,\nu^h_{2s}\right).$$

This states that the agent purchases enough assets or promises paying in good 2, $\theta^h_{2s},\sigma^h_{2s},\nu^h_{2s}$, to back up her own asset-backed securities or issued promises σ^h_{1s},ν^h_{2s}. The above condition can be rearranged as

$$p(z_s)\max\left(0,\psi^h_{2s}\right)+p(z_s)\max\left(0,\sigma^h_{2s}\right)+p(z_s)\nu^h_{2s} \quad \geq \quad -\min\left(0,\sigma^h_{1s}\right), \tag{A.161}$$

where we applies the fact that $\max\left(0,\nu^h_{2s}\right)+\min\left(0,\nu^h_{2s}\right)=\nu^h_{2s}$.

Similarly, the collateral requirement condition for issued securities that require financial assets paying in good 1 as collateral is given by

$$\max\left(0,\psi^h_{1s}\right)+\max\left(0,\sigma^h_{1s}\right)+\nu^h_{1s} \quad \geq \quad -p(z_s)\min\left(0,\sigma^h_{2s}\right),\ \forall s, \tag{A.162}$$

where the right-hand-side comes from the fourth and fifth rows of Table A.11.

We now show that the collateral constraints

$$p(z_s)R_sk^h+\theta^h_{1s}+p(z_s)\theta^h_{2s} \quad \geq \quad 0,\ \forall s \tag{A.163}$$

are equivalent to collateral requirement conditions (with three types of collateral), (A.160), (A.161), and (A.162). In other words, there is no loss of generality to use the collateral constraints (A.163); an allocation is attainable under (A.163) if and only if it is so under (A.160), (A.161), and (A.162).

To be more precise, let $\theta^h_{1s}=\psi^h_{1s}+\sigma^h_{1s}+\nu^h_{1s}$ and $\theta^h_{2s}=\psi^h_{2s}+\sigma^h_{2s}+\nu^h_{2s}$ be contingent securities paying in good 1 and in good 2 in state s, respectively, which can be backed either by good 2 or purchased assets (other people's promises). Note that θ^h_{1s} and θ^h_{2s} include both directly collateralized and asset-backed securities. An attainable allocation under (A.160), (A.161), and (A.162) can be defined similarly to the one under (A.87) by replacing (A.92) the following resource constraints:

$$\sum_h \alpha^h \psi_{1s}^h = \sum_h \alpha^h \psi_{2s}^h = \sum_h \alpha^h \sigma_{1s}^h = \sum_h \alpha^h \sigma_{2s}^h = \sum_h \alpha^h \nu_{1s}^h = \sum_h \alpha^h \nu_{2s}^h = 0, \ \forall s. \quad \text{(A.164)}$$

The collateral constraint (A.163) results from summing (A.160), (A.161), and (A.162) altogether, and then applying $\max(0, x) + \min(0, x) = x$ to get rid of max and min operators. In addition, the proof of this lemma also shows how to recover contract allocation $\left(\psi_{1s}^h, \psi_{2s}^h, \sigma_{1s}^h, \sigma_{2s}^h, \nu_{1s}^h, \nu_{2s}^h\right)_h$ from $\left(\theta_{1s}^h, \theta_{2s}^h\right)$.

Lemma A.3. *The following statements are true:*

(i) if $\left(\mathbf{c}_0^h, k^h, \psi_{1s}^h, \psi_{2s}^h, \sigma_{1s}^h, \sigma_{2s}^h, \nu_{1s}^h, \nu_{2s}^h\right)_h$ is attainable, then the collateral constraint (A.163) and the market-clearing conditions (A.92) hold, and

(ii) if $\left(k^h, \theta_{1s}^h, \theta_{2s}^h\right)_h$ is attainable, then there exists a collateral and security allocation $\left(k^h, \psi_{1s}^h, \psi_{2s}^h, \sigma_{1s}^h, \sigma_{2s}^h, \nu_{1s}^h, \nu_{2s}^h\right)_h$ that satisfies collateral requirement conditions (A.160), (A.161), (A.162) and the market-clearing conditions (A.164).

Proof. The first statement can be proved as follows. First, it is clear that conditions (A.164) imply (A.92). We now only need to show that (A.160), (A.161), and (A.162) imply (A.163). Summing up all collateral requirement conditions, (A.160), (A.161), and (A.162), and using the fact that $\max(0, x) + \min(0, x) = x$ give, for an agent h in state s,

$$p(z_s)R_s k^h + \left[\psi_{1s}^h + \sigma_{1s}^h + \nu_{1s}^h\right] + p(z_s)\left[\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right] \quad \geq \quad 0,$$

which is the collateral constraint for an agent h in state s where $\theta_{1s}^h = \psi_{1s}^h + \sigma_{1s}^h + \nu_{1s}^h$ and $\theta_{2s}^h = \psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h$.

The second statement is proved as follows. Consider an allocation $\left(k^h, \theta_{1s}^h, \theta_{2s}^h\right)_h$ that satisfies (A.163) and (A.92). We choose a corresponding allocation $\left(k^h, \psi_{1s}^h, \psi_{2s}^h, \sigma_{1s}^h, \sigma_{2s}^h, \nu_{1s}^h, \nu_{2s}^h\right)_h$ that satisfies $\theta_{1s}^h = \psi_{1s}^h + \sigma_{1s}^h + \nu_{1s}^h$, $\theta_{2s}^h = \psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h$, the collateral requirement conditions (A.160), (A.161), (A.162), and the market-clearing conditions (A.164). Consider the following candidate allocation:

$$\psi_{1s}^h = \theta_{1s}^h + p(z_s)\theta_{2s}^h, \quad \text{(A.165)}$$

$$\psi_{2s}^h = \nu_{1s}^h = \nu_{2s}^h = 0, \quad \text{(A.166)}$$

$$\sigma_{1s}^h = \theta_{1s}^h - \psi_{1s}^h = -p(z_s)\theta_{2s}^h, \quad \text{(A.167)}$$

$$\sigma_{2s}^h = \theta_{2s}^h. \quad \text{(A.168)}$$

(A.166) implies that agents hold no $\psi^h_{2s}, \nu^h_{1s}, \nu^h_{2s}$; they will borrow or lend through directly collateralized contract paying in good 1 ψ^h_{1s} only.

It is straightforward to show that resource constraints (A.164) hold. Since the resource constraints are satisfied and the collateral allocations k^h are the same, the market fundamentals are the same. We now would like to show that collateral requirement conditions (A.160), (A.161), (A.162) also hold. First, we will show that (A.161) and (A.162) hold. There are two cases to consider; (i) $\theta^h_{2s} > 0$, (ii) $\theta^h_{2s} < 0$. Case I: Suppose that $\theta^h_{2s} > 0$. Using (A.168), this implies that $\sigma^h_{2s} > 0$, which in turn leads to $\min\left(0, \sigma^h_{2s}\right) = 0$. On the other hand, it is true that

$$\max\left(0, \psi^h_{1s}\right) + \max\left(0, \sigma^h_{1s}\right) = \max\left(0, \psi^h_{1s}\right) + \max\left(0, \sigma^h_{1s}\right) + \nu^h_{1s} \geq 0,$$

where the first equality follows from (A.166). Since $\min\left(0, \sigma^h_{2s}\right) = 0$, we have

$$\max\left(0, \psi^h_{1s}\right) + \max\left(0, \sigma^h_{1s}\right) = \max\left(0, \psi^h_{1s}\right) + \max\left(0, \sigma^h_{1s}\right) + \nu^h_{1s} \geq -p(z_s)\min\left(0, \sigma^h_{2s}\right),$$

which is (A.162). On the other hand, (A.167) implies that $\sigma^h_{1s} < 0$ when $\theta^h_{2s} > 0$. As a result, $\min\left(0, \sigma^h_{1s}\right) = \sigma^h_{1s}$. Using (A.166), (A.167), (A.168), we then can show that

$$\begin{aligned} &p(z_s)\max\left(0, \psi^h_{2s}\right) + p(z_s)\max\left(0, \sigma^h_{2s}\right) + p(z_s)\nu^h_{2s} + \min\left(0, \sigma^h_{1s}\right) \\ &\qquad = 0 + p(z_s)\sigma^h_{2s} + 0 + \sigma^h_{1s} = p(z_s)\theta^h_{2s} - p(z_s)\theta^h_{2s} = 0, \end{aligned}$$

where the second equality follows from (A.167) and (A.168). This shows that (A.161) holds.

Case II: Suppose that $\theta^h_{2s} < 0$. (A.167) and (A.168) imply that $\max\left(0, \sigma^h_{1s}\right) = \sigma^h_{1s} = -p(z_s)\theta^h_{2s}$ and $\min\left(0, \sigma^h_{2s}\right) = \sigma^h_{2s} = \theta^h_{2s}$, respectively. We then can write

$$\max\left(0, \psi^h_{1s}\right) + \max\left(0, \sigma^h_{1s}\right) + \nu^h_{1s} = \max\left(0, \psi^h_{1s}\right) - p(z_s)\theta^h_{2s} \geq -p(z_s)\theta^h_{2s} = -p(z_s)\min\left(0, \sigma^h_{2s}\right),$$

which is exactly (A.162). Note that the first equality follows from (A.166), the second inequality follows from the fact that $\max\left(0, \psi^h_{1s}\right) \geq 0$. Similarly, using , we can show that $\max\left(0, \sigma^h_{2s}\right) = \min\left(0, \sigma^h_{1s}\right) = 0$. This implies that

$$p(z_s)\max\left(0, \psi^h_{2s}\right) + p(z_s)\max\left(0, \sigma^h_{2s}\right) + p(z_s)\nu^h_{2s} + \min\left(0, \sigma^h_{1s}\right) = 0 + 0 + 0 + 0 = 0,$$

which is exactly (A.161).

Similarly, we can now show that (A.160) also holds. There are two cases to be considered as well.

Case I: suppose that $\theta_{1s}^h + p(z_s)\theta_{2s}^h < 0$. (A.165) implies that $\psi_{1s}^h < 0$, which in turn implies that $\min\left(0, \psi_{1s}^h\right) = \psi_{1s}^h = \theta_{1s}^h + p(z_s)\theta_{2s}^h$. Using (A.166), we now can show that

$$p(z_s)R_s k^h + \min\left(0, \psi_{1s}^h\right) + p(z_s)\min\left(0, \psi_{2s}^h\right) = p(z_s)R_s k^h + \theta_{1s}^h + p(z_s)\theta_{2s}^h + 0 \geq 0,$$

where the last inequality follows (A.163). This implies that (A.160) holds.

Case II: we can use a similar argument to show that (A.160) holds when $\theta_{1s}^h + p(z_s)\theta_{2s}^h = \psi_{1s}^h > 0$. In summary, we have show that all collateral requirement conditions hold. □

F.6.2 Pooling Collateral versus Tranching

This section shows that the markets economize on collateral; that is, there is no gain from pooling collateral across agents type h. Let the collateral constraints with pooling be:

$$p(z_s)R_s K \geq -\sum_h \alpha^h p(z_s)\min\left\{0, \psi_{2s}^h\right\} - \sum_h \alpha^h \min\left\{0, \psi_{1s}^h\right\}, \tag{A.169}$$

where the average collateral $K = \sum_h \alpha^h k^h$. We then show that the group collateral constraint is equivalent to individuals collateral constraints (A.95).

Lemma A.4. *For any allocation $\left(k^h, \psi_{2s}^h, \tau_{2s}^h, \tau_{1s}^h\right)$ satisfying the collateral constraints (A.169), then there exists there exists an equivalent attainable allocation $\left(k'^h, \psi_{2s}'^h, \tau_{2s}^h, \tau_{1s}^h\right)$ with*

$$k'^1 = \frac{\sum_h \alpha^h k_s^h}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1, \tag{A.170}$$

$$\psi_{2s}'^h = \left(R_s k_s^h + \psi_{2s}^h\right) - R_s k'^h, \tag{A.171}$$

where $k_s^h = \frac{-p(z_s)\min\left(0, \psi_{2s}'^h\right) - \min\left(0, \psi_{1s}'^h\right)}{p(z_s)R_s}, \forall s.$

Proof. This result can be proved in two steps: (i) show that the collateral constraints (A.169) hold if and only if there exists k_s^h such that (A.95) hold, (ii) then show that any allocation with state-contingent collateral, k_s^h, can be replicated by an allocation with fixed collateral allocation k^h.

Step I: $\Longrightarrow$ Suppose that collateral constraints (A.169) hold. Now consider an alternative allocation with

$$k_s^h = \frac{-p(z_s)\min\left(0, \psi_{2s}'^h\right) - \min\left(0, \psi_{1s}'^h\right)}{p(z_s)R_s}, \forall s. \tag{A.172}$$

This clearly implies no default. We then only need to show that the average collateral needed $\sum_h \alpha^h k_s^h$ is no larger than K. Summing the above equation over h with weight α^h gives, for each s,

$$\begin{aligned} \sum_h \alpha^h k_s^h &= \sum_h \alpha^h \frac{-p(z_s)\min\left(0, \psi_{2s}'^h\right) - \min\left(0, \psi_{1s}'^h\right)}{p(z_s) R_s} \\ &\leq K, \end{aligned} \tag{A.173}$$

where the last inequality follows from the group collateral constraints (A.169).

$\Longleftarrow$ This can be done by summing over the individuals collateral constraints with weight α^h.

Step II: Let $\left(k_s^h, \psi_{2s}^h, \tau_{2s}^h, \tau_{1s}^h\right)$ be an attainable allocation with contingent collateral; that is, it satisfies the collateral constraint for each h and s:

$$R_s k_s^h \geq -\min\left(0, \psi_{2s}'^h\right), \tag{A.174}$$

and the average collateral is the same in every state; $K = \sum_h \alpha^h k_s^h$ for all s. In addition, the consumption allocation of agent h in state s is given by

$$c_{1s}^h = e_{1s}^h + \tau_{1s}^h, \tag{A.175}$$

$$c_{2s}^h = e_{2s}^h + \left(R_s k_s^h + \psi_{2s}^h\right) + \tau_{2s}^h, \tag{A.176}$$

where the spot trade satisfies:

$$\tau_{1s}^h + p(z_s)\tau_{2s}^h = 0. \tag{A.177}$$

Now consider a candidate allocation $\left(k'^h, \psi_{2s}'^h, \tau_{2s}'^h, \tau_{1s}'^h\right)$ with

$$k'^1 = \frac{K}{\alpha^1}, \text{ and } k'^h = 0 \text{ for } h \neq 1, \tag{A.178}$$

$$\psi_{2s}'^h = \left(R_s k_s^h + \psi_{2s}^h\right) - R_s k'^h, \tag{A.179}$$

$$\tau_{1s}'^h = \tau_{1s}^h, \text{ and } \tau_{2s}'^h = \tau_{2s}^h. \tag{A.180}$$

Using (A.178), we can write the securities as

$$\psi_{2s}'^1 = \left(R_s k_s^1 + \psi_{2s}^1\right) - k'^1, \tag{A.181}$$

$$\psi_{2s}'^h = R_s k_s^h + \psi_{2s}^h \text{ for } h \neq 1. \tag{A.182}$$

Using the collateral constraint (A.174) we can show that for each $h \neq 1$:

$$\psi_{2s}'^h = R_s k_s^h + \psi_{2s}^h \geq R_s k_s^h + \min\left\{0, \psi_{2s}^h\right\} \geq 0, \tag{A.183}$$

where the last inequality follows from the collateral constraint (A.174). This, $\psi_{2s}'^h \geq 0$, implies that the collateral constraint for any $h \neq 1$ holds (since he does not issue securities at all).

We hence only need to show that the collateral constraint also holds for $h = 1$. We can rewrite (A.181) as

$$k'^1 = \left(R_s k_s^1 + \psi_s^1\right) - \psi_s'^1 \geq -\psi_s'^1, \tag{A.184}$$

where the last inequality follows from the collateral constraint (A.174) for $h = 1$. This shows that the collateral constraint also holds for $h = 1$.

Given that $\sum_h \alpha^h k_s'^h = K = \sum_h \alpha^h k_s^h$, the market fundamentals are the same for every state. With the same market fundamental, z_s, the spot trade is satisfied, using (A.180).

Now we will show that the consumption allocations are also the same.

$$c_{1s}'^h = e_{1s}^h + \tau_{1s}'^h = e_{1s}^h + \tau_{1s}^h = c_{1s}^h, \tag{A.185}$$

where the second equality follows from (A.180), and the last one follows from (A.175). Similarly,

$$\begin{aligned} c_{2s}'^h &= e_{2s}^h + \left(R_s k_s'^h + \psi_{2s}'^h\right) + \tau_{2s}'^h = e_{2s}^h + \left(R_s k_s'^h + \left(R_s k_s^h + \psi_{2s}^h - R_s k'^h\right)\right) + \tau_{2s}^h \\ &= e_{2s}^h + \left(R_s k_s^h + \psi_{2s}^h\right) + \tau_{2s}^h = c_{2s}^h, \end{aligned} \tag{A.186}$$

where the second equality follows from (A.179) and (A.180), and the last one follows from (A.176).

□

F.6.3 Ex-ante Contracting versus Ex-post Spot Trading

Thus far we implicitly shut down trade in the spot markets in each state. This section shows that the spot markets are redundant when all types of contracts are available (see Lemma A.5 below). In other words, agents do not need to trade in spot markets, though they may well do so. Importantly, the spot markets are open and deliver the spot price $p(z_s)$. In addition, we also show that the asset-backed securities are not necessary when the spot markets are open and active (see Lemma A.6 below). Put differently, agents simply are indifferent between trading in spot markets or ex-ante asset-backed securities.

When the spot markets are open, each agent h can trade τ_{1s}^h units of good 1 for τ_{2s}^h units of good 2 at a spot price $p(z_s)$ according to the spot-trade constraint:

$$\tau_{1s}^h + p(z_s)\tau_{2s}^h = 0. \tag{A.187}$$

Recall that the spot price function, $p(z_s)$, is the price such that the spot markets for both goods clear:

$$\sum_h \alpha^h \tau_{1s}^h = 0, \tag{A.188}$$

$$\sum_h \alpha^h \tau_{2s}^h = 0. \tag{A.189}$$

Hence, an attainable allocation with the spot markets is defined by adding the spot-trade constraint (A.187) and market-clearing constraints (A.188)-(A.189) to Definition A.7.

To be more precise, an allocation is said to be *equivalent* to an attainable allocation if it is attainable and generates the same consumption allocation and market fundamental in each state s as the original attainable allocation.

Lemma A.5. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}^h, \tau_{\ell s}^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ *such that*

$$\tau_{\ell s}'^h = 0, \forall s, h, \ell. \tag{A.190}$$

Proof. Let $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}^h, \tau_{\ell s}^h\right)_h$ be an attainable allocation. We will show that we can find an equivalent allocation with no spot trade, i.e., $\tau_{\ell s}'^h = 0$. Consider the following candidate allocation (with $'$)

$$\mathbf{c}_0'^h = \mathbf{c}_0^h, \forall h, \tag{A.191}$$

$$\theta_{1s}'^h = \theta_{1s}^h + \tau_{1s}^h, \forall s, h, \tag{A.192}$$

$$\theta_{2s}'^h = \theta_{2s}^h + \tau_{2s}^h, \forall s, h. \tag{A.193}$$

Note that agents here acquire or issue securities on good 1 and good 2 in state s rather than waiting for trade in spot markets. The rest of the proof is similar to the proof of Lemma A.3, and hence is omitted. □

Condition (A.190) in Lemma A.5 implies that the spot markets in period 1 are redundant when all securities are allowed; that is, anything that can be done through the spot markets and one set of securities is feasible under another set of securities without spot markets. Henceforth (and previously), the ex-post spot trade transfers will be (were) set to zero, ($\tau_{\ell s}^h = 0$ as in (A.190)) and the spot-trade constraints (A.187) will be (were) neglected, unless stated otherwise.

Lemma A.6. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}^h, \sigma_{\ell s}^h, \nu_{\ell s}^h, \tau_{\ell s}^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}'^h, \sigma_{\ell s}'^h, \nu_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ *such that*

$$\sigma_{1s}'^h = \sigma_{2s}'^h = \nu_{1s}'^h = \nu_{2s}'^h = 0, \forall s, h. \tag{A.194}$$

Proof. Suppose $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}^h, \sigma_{\ell s}^h, \nu_{\ell s}^h, \tau_{\ell s}^h\right)_h$ is attainable. Consider the following alternative allocation (with $'$) $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}'^h, \sigma_{\ell s}'^h, \nu_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ such that for all h and all s

$$\sigma_{1s}'^h = \sigma_{2s}'^h = \nu_{1s}'^h = \nu_{2s}'^h = \psi_{2s}^h = 0, \tag{A.195}$$

$$\psi_{1s}'^h = \left(\psi_{1s}^h + \sigma_{1s}^h + \nu_{1s}^h\right) + p(z_s)\left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right), \tag{A.196}$$

$$\tau_{1s}'^h = -p(z_s)\left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right) + \tau_{1s}^h, \tag{A.197}$$

$$\tau_{2s}'^h = \left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right) + \tau_{2s}^h. \tag{A.198}$$

Note that at the alternative allocation, agents will do in spot markets what they might have done in asset-backed security markets. In addition, with active spot markets, there is no need to trade in collateral-backed securities paying in good 2 (trade in the ones paying in numeraire good only). The rest of the proof is similar to the proof of Lemma A.5, and hence is omitted. □

It is worthy of emphasis that Lemma A.5 and Lemma A.6 imply that the asset-backed securities that we need in this model are the ones that replicate spot markets. In other words, the asset-backed securities in this model (with tranching) are simply substitutes for spot markets. Henceforth, we let asset-backed securities play this role and shut down active trade in spot markets. The result is summarized in the following corollary.

Corollary A.1. *Asset-backed securities and the spot markets are perfect substitute in this model.*

F.6.4 Spot Markets and Security Prices: No-Arbitrage Condition

The pyramiding mechanism puts a restriction on the prices of contracts traded within each security exchange. The ratio of the equilibrium prices of the securities in security exchange z_s in state s, $\frac{Q_{2s}}{Q_{1s}}$, must be equal to the marginal rate of substitution or the spot price in the security exchange, $p(z_s)$. Otherwise, there will be an arbitrage possibility (by keeping the collateral constraints satisfied with pyramiding). The result is summarized in the following lemma.

Lemma A.7. *In a competitive equilibrium, for each s and z_s,*

$$Q_{2s} = p(z_s)Q_{1s}. \tag{A.199}$$

Using the no-arbitrage condition (A.199), the collateral constraints (A.163) can be rewritten as

$$Q_{2s}R_s k^h + Q_{1s}\theta_{1s}^h + Q_{2s}\theta_{2s}^h \geq 0, \forall s. \tag{A.200}$$

These constraints state that the value in units of good 1 at $t = 0$ of all ex ante securities held (RHS) cannot exceed the value of collateral held (LHS). These constraints are applicable when the spot markets are not available but the ex-ante asset-backed securities can be traded.

G A General Model with Price Externalities and Its Prototypical Economies

This section formulates a general model that captures key features regarding price externalities of 6 prototypical economies including a collateral economy (Kilenthong and Townsend, 2014), an exogenous incomplete markets economy (Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986), a moral hazard with retrading economy (Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011), a liquidity constrained economy (Hart and Zingales, 2013), a fire sales economy (Lorenzoni, 2008), and a hidden information with retrading economy (Diamond and Dybvig, 1983; Jacklin, 1987). Each subsection presents a key ingredient of the model along with the relevant part of each prototypical economy.

G.1 Basic Ingredients: Commodity Space, Preferences, Endowments, and Technology

There are L commodities. These can be basic underlying commodities and also date and/or state contingent where the date and/or state are public. In order to incorporate private information problems into this framework, we also allow a subset of commodities to be contingent on recommended but unobserved actions or on reported but unobserved states. For actions, let a be the recommended action and (with the incentive compatibility constraints in place) the actually taken action, and a' be potentially deviating action. For privately observed states, let a be the reported state and (with incentive compatibility constraints in place) the actual state, and let a' be some potentially counterfactual report. Let $A \in \mathbb{R}_+$ be the set of possible actions/states, i.e., $a, a' \in A$.

There is a continuum of agents of measure one. The agents are divided into H (ex-ante) types, each of which is indexed by $h = 1, 2, \cdots, H$. Each type h consists of $\alpha^h \in [0, 1]$ fraction of the population such that $\sum_h \alpha^h = 1$. In addition, this model allows for ex-post diversity denoted by ex-post (either observable or unobservable) type $\omega \in \Omega$. More formally, let $\zeta^h(\omega)$ be the fraction of agents of type h whose ex-post type is ω. An ex-post type ω may depend on an observed output, an unobserved action, and/or unobserved state of nature as well.

Each agent type h is endowed with an endowment $\mathbf{e}^h \in \mathbb{R}^L_+$. Note that $\mathbf{c}^h$ and $\mathbf{e}^h$ lie in the L-dimensional commodity space. The preferences of an agent of type h are represented by the utility function $U^h\left(\mathbf{c}^h\right)$, where $\mathbf{c}^h \in \mathbb{R}^L$ is the consumption allocation for an agent of type h.

Each agent of type h has an access to a production technology defined implicitly by

$$F^h\left(\mathbf{y}^h\right) \geq 0, \tag{A.201}$$

where $\mathbf{y}^h \in \mathbb{R}^L$ is the vector of its inputs and outputs in commodity space L. This production technology is generally a multidimensional vector of constraints with dimension O, i.e., $F^h\left(\mathbf{y}^h\right) \equiv \left[F^h_o\left(\mathbf{y}^h\right)\right]_{o=1}^{O}$.

G.1.1 Basic Ingredients for the Collateral Economy

This is a two-period economy, $t = 0, 1$. There are a finite S states of nature in the second period $t = 1$, i.e., $s = 1, 2, ..., S$. Let $0 < \pi_s \leq 1$ be the objective and commonly assessed probability of state s occurring, where $\sum_s \pi_s = 1$. There are two goods, called good 1 and good 2 in each period. These two goods can be traded in each date and in each state, and we refer to those markets as spot markets with good 1 as the numeraire good in every date and state. Thus, there are $L = 2(1+S)$ commodities. There is no unobserved action or privately observed state.

Each agent of type $h = 1, 2, \ldots, H$ is endowed with good 1 and good 2, $\mathbf{e}^h_0 = \left(e^h_{10}, e^h_{20}\right)$ in the first period and $\mathbf{e}^h_s = \left(e^h_{1s}, e^h_{2s}\right)$, in each state $s = 1, \cdots, S$. Let $\mathbf{e}^h = \left(\mathbf{e}^h_0, \cdots, \mathbf{e}^h_S\right)$ be the endowment profile of an agent of type h over the first period and all states s in the second period, respectively. There is no ex-post diversity in this economy, and therefore we simply omit all related notation.

The preferences of an agent of type h are represented by the utility function $u^h\left(c^h_1, c^h_2\right)$, and the discounted expected utility of h is defined by:

$$U^h\left(\mathbf{c}^h\right) \equiv u^h\left(c^h_{10}, c^h_{20}\right) + \beta \sum_{s=1}^{S} \pi_s u^h\left(c^h_{1s}, c^h_{2s}\right), \tag{A.202}$$

where β is the discount factor.

Good 1 is consumable but cannot be stored from $t = 0$ to $t = 1$ (is completely perishable), while good 2 is consumable and storable. The good 2 that is stored can be collateralizable, i.e., can serve as collateral to back promises. Henceforth, good 2 and the collateral good will be used interchangeably. Each unit of good 2 stored (as input) will become R_s units of good 2 in state s.

As a result, the production function in our general framework can be written as follows:

$$F_s^h(\mathbf{y}^h) = -y_{2s}^h - R_s y_{20}^h \geq 0, \text{for } s = 1, \ldots, S, \tag{A.203}$$

where $y_{20}^h \in \mathbb{R}_-$ and $y_{2s}^h \in \mathbb{R}_+, s = 1, 2, \ldots, S$ are inputs and outputs, respectively. We use the standard convention under which an input must be non-positive and an output must be non-negative. This economy has $O = S$ production functions.

G.1.2 Basic Ingredients for the Exogenous Incomplete Markets Economy

Consider an economy with two periods, $t = 0, 1$. There are S possible states of nature in the second period $t = 1$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s such that $\sum_s \pi_s = 1$. There are 2 goods, labeled good 1 and good 2, in each date and in each state. Thus, there are $L = 2(1 + S)$ commodities. Because the endowment profiles are the same as specified in the collateral economy discussed above, we omit the details in this section for brevity.

The preferences of an agent of type h are represented by the utility function $u^h\left(c_1^h, c_2^h\right)$, and the discounted expected utility of h is defined by:

$$U^h\left(\mathbf{c}^h\right) \equiv u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u^h\left(c_{1s}^h, c_{2s}^h\right), \tag{A.204}$$

where β is the discount factor. There is no ex-post diversity in this economy, and endowments and preferences are known ex-ante, and therefore we simply omit all related notation.

For simplicity, we assume that there is no production. Thus, F_o^h can be suppressed. As a result, there would be no externalities if preferences were identically homothetic, as spot prices are determined by ratio of aggregate endowment only, which no one can influence. So we assume otherwise; that is, preferences are not identically homothetic.

G.1.3 Basic Ingredients for the Moral Hazard with Retrading Economy

There are two physical commodities, labeled as good 1 and good 2, in each states. These commodities can be produced using the sole input, called action, a. Let A be the number of possible actions. As in the literature, the random production technology is given by $f(\mathbf{q}|a)$, which is the probability density function of the output vector of good 1 and good 2, $\mathbf{q} = (q_1, q_2)$, conditional on an action a taken by an agent. In other words, the probability that the realized output will be $\mathbf{q}$ is $f(\mathbf{q}|a)$ when an agent takes an action a. The action that an agent takes is *private information*. Hence, there is a *moral hazard* problem. There is a continuum of ex ante identical agents of mass 1, i.e., no diversity in types so trivially $\alpha^1 = 1$. For simplicity, we assume that each agent is endowed with zero units of both goods.

We will now map this moral hazard economy into our general model with securities trading. Different combinations of outputs $\mathbf{q}$ define (idiosyncratic) states or indexes for contracting purposes. There is no loss of generality to assume that there are a finite Q states, $\mathbf{q} \in Q$. Following the mechanism design literature, an optimal consumption of the two goods under moral hazard depends on realized output $\mathbf{q}$ and recommended action a; that is, $c_1(\mathbf{q}, a)$ and $c_2(\mathbf{q}, a)$. Accordingly, we define commodity using both output/state $\mathbf{q}$ and recommended action a. In particular, for each recommended action a, there are Q states. There are two commodities in each state. In addition, actual action a itself is another commodity. Therefore, there are $L = 2QA + 1$ commodities in this model.

Each agent is endowed with the instantaneous common utility function for the two goods and action, $u(c_1, c_2, a)$. Again, let a be recommended action, and a' be taken (possibly out-of equilibrium) action. The discounted expected utility of an agent who is reported action a but took action a' is defined by:

$$U(\mathbf{c}) = \sum_{\mathbf{q}} \pi(\mathbf{q}|a') u\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a'\right) \tag{A.205}$$

where $\pi(\mathbf{q}|a)$ denote the probability of realizing outputs $\mathbf{q}$ given action a (actually taken), which satisfies the following probability constraint:

$$\sum_{\mathbf{q}} \pi(\mathbf{q}|a) = 1, \forall a. \tag{A.206}$$

Ex-post diversity in this model is determined by actual (ex ante) action and realized (ex post) outputs, i.e., $\omega = (\mathbf{q}, a')$. For generality, let $\delta(a')$ be the fraction of agents who took action a'. Recall that the fraction of agents who realized outputs $\mathbf{q}$ conditional on taking action a' is $f(\mathbf{q}|a')$. As a result, the fractions of agents of ex-post type $(\mathbf{q}, a')$ is $\zeta^1(\mathbf{q}, a') = f(\mathbf{q}|a')\,\delta(a')$.

As in the literature, the probability distribution across outputs/states depend on agent's choice of action a. This dependency is modeled as a general production function F whose input is actual action a and outputs are $\mathbf{q}$:

$$F(\mathbf{q}, a) = f(\mathbf{q}|a) - \pi(\mathbf{q}|a) = 0, \forall \mathbf{q}, a \tag{A.207}$$

In words, different actions will lead to different probability distributions. There are, as in (A.207), $O = QA$ production functions. Combining these production technologies with the probability conditions (A.206) leads to standard probability constraints of production function $f(\mathbf{q}|a)$:

$$\sum_{\mathbf{q}} \pi(\mathbf{q}|a) = 1 \Rightarrow \sum_{\mathbf{q}} f(\mathbf{q}|a) = 1, \forall a. \tag{A.208}$$

G.1.4 Basic Ingredients for the Liquidity Constrained Economy

Consider an economy with four periods, $t = 0, 1, 2, 3$. There are two types of agents, called "doctors" and "builders", each of which consists of $\alpha^h > 0$ for all $h = b, d$ fraction of the population with $\sum_{h=b,d} \alpha^h = 1$. Each agent $h = b, d$ is endowed with $e^h = e$ units of wheat at period $t = 0$. This is a simplified and deterministic version of Hart and Zingales (2013) in which we assume that the doctors will buy building services in period $t = 1$ first, and the builders will buy doctor services later in period $t = 2$. As in the collateral model in section G.1.1, there is no unobserved action or privately observed state, and therefore we simply omit all related notations.

There are two commodities in period $t = 0$, wheat w_0^h, and storage f_0^h, where the latter is formally defined below. There are three commodities in period $t = 1$, storage f_1^h, building services b^d and labor supply of the doctors l^d. Similarly, there are three commodities in period $t = 2$, storage f_2^h, doctor services d^b, and labor supply of the builders l^b. There is one commodity, wheat w_3^h, in the last period $t = 3$. Therefore, there are $L = 9$ commodities in this model.

The preferences of doctors and builders are represented by

$$U^d(\mathbf{c}) = u^d\left(w^d, d^d, b^d, l^d\right) = w_3^d + b^d - \frac{\left(l^d\right)^2}{2}, \tag{A.209}$$

$$U^b(\mathbf{c}) = u^b\left(w^b, d^b, b^b, l^b\right) = w_3^b + d^b - \frac{\left(l^b\right)^2}{2}, \tag{A.210}$$

respectively. Note that doctors do not consume doctor services, and vice versa for builders. We can write the utility function in a more general from as follows:

$$U^h(\mathbf{c}) = u^h\left(w^h, d^h, b^h, l^h\right) = w_3^h + \delta_b^h b^h + \left(1 - \delta_d^h\right) d^h - \frac{\left(l^h\right)^2}{2}, \tag{A.211}$$

where $\delta_b^h = 1$ if $h \neq b$, and zero otherwise.

There are two technologies or assets available in period $t = 0$. First, the collateralizable asset is a storage technology, whose return from $t = 0$ to $t = 3$ is 1 unit of wheat, i.e., saving one unit of wheat in the first period $t = 0$ will return 1 unit of wheat in the last period $t = 3$. In addition, the claim on the output of this technology is transferable, and therefore can be used as private money (or collateral) during periods $t = 1$ and $t = 2$. The second asset is an investment project, whose return from $t = 0$ to $t = 3$ is $\bar{R} > 1$ units of wheat. However, this asset cannot be used as collateral.

For simplicity, we consider only a deterministic return case here. Let f_0^h be the amount of wheat stored by an agent type $h = b, d$, and accordingly, the agent type h invests $e - f_0^h$ units of wheat in the investment project.

The production technologies are irreversible; that is, their outputs will be realized in the last period $t = 3$ only. The production function of the storage technology (denoted by subscript "s") for an agent type h is defined as follows:

$$F_s^h\left(f_2^h, y_{31}^h\right) = y_{s3}^h - f_2^h = 0, \forall h = b, d, \tag{A.212}$$

where f_2^h is the number of claims on the storage technology held by the agent type h at the end of period $t = 2$, and y_{s3}^h is the output in unit of wheat in period $t = 3$ received by the agent type h from the storage technology. Similarly, the production function for investment project (denoted by subscript "i") is defined by

$$F_i^h\left(e - f_0^h, y_{i3}^h\right) = y_{i3}^h - \bar{R}\left(e - f_0^h\right) = 0, \forall h, \tag{A.213}$$

where f_0^h is the amount of wheat stored by an agent type h in period $t = 0$, and y_{i3}^h is the output in unit of wheat in period $t = 3$ received by the agent type h from the investment technology .

In addition, the builders and the doctors produce building and doctor services (denoted by subscript "o"), respectively, using the following simple linear technologies:

$$F_o^h\left(y_h^h, l^h\right) = y_h^h - l^h = 0, \forall h = b, d, \tag{A.214}$$

which use labor as the only input. For notational convenience, we also set

$$F_0^h\left(y_{-h}^h\right) = y_{-h}^h = 0, \forall h = b, d, \tag{A.215}$$

where $y_{-h}^h = y_b^d, y_d^b$ denote building services produced by doctors and vice versa. To sum up, there are $O = 4$ production functions.

G.1.5 Basic Ingredients for the Fire Sales Economy

Consider an economy with three periods, $t = 0, 1, 2$. There are two states, $s = 1, 2$, realized in period $t = 1$, with probability π_1 and π_2, respectively. We use histories of these states to define states in period $t = 2$; that is, if the state $s = 1$ is realized in period $t = 1$, then the state in period $t = 2$ will automatically be $s = 1$. Therefore, there are two states $s = 1, 2$ in the last period $t = 2$. There are two types of agents, called "consumers" and "entrepreneurs", each of which consists of equal mass. A consumer receives an endowment of e units of consumption goods in each period while an entrepreneur is endowed with n units of consumption goods in the first period $t = 0$ only.

There are $L = 8$ commodities in this model, i.e., two physical goods, namely consumption and capital goods, in period $t = 0$, two physical goods, namely consumption and capital goods, at each state $s = 1, 2$ in period $t = 1$, and one physical good, namely consumption good, at each state $s = 1, 2$ in period $t = 2$.

The preferences of a consumer is represented by

$$U^c(\mathbf{c}) = E\left[c_0^c + c_1^c + c_2^c\right] = c_0^c + \sum_{s=1,2} \pi_s \left(c_{1s}^c + c_{2s}^c\right), \tag{A.216}$$

where superscript "c" stands for consumer, c_0^c is the consumer's consumption in period 0, and c_{ts}^c is the consumer's consumption at state $s = 1, 2$ in period $t = 1, 2$. The preferences of an entrepreneur is represented by

$$U^e(\mathbf{c}) = E\left[c_2^e\right] = \sum_{s=1,2} \pi_s c_{2s}^e, \tag{A.217}$$

where superscript "e" stands for entrepreneur, c_{2s}^e is the entrepreneur's consumption at state $s = 1, 2$ in period $t = 2$.

The following production functions, and inputs/outputs are generally written as $F_{\ell ts}^h$, and $y_{\ell ts}^h$, where $h = c, e$ denotes an agent type, $\ell = c, i, k, n, o, p, r$ denotes an input/output type of commodities ("c" stands for consumption goods, "i" stands for input for new capital production, "k" stands for capital input, "n" stands for new capital, "o" stands for old capital, "r" stands for repairing input for capital maintenance) or a constraint type ("p" stands for weakly positive constraints), $t = 0, 1, 2$ denotes a period, and $s = 0, 1, 2$ denotes a state with state $s = 0$ for period $t = 0$.

Each entrepreneur can turn a unit of consumption good into a unit of (new) capital good at any period and any state of nature. This constitutes the first set of production functions:

$$F_{n00}^e\left(y_{n00}^e, y_{i00}^e\right) = y_{n00}^e + y_{i00}^e = 0, \tag{A.218}$$

$$F_{n1s}^e\left(y_{n1s}^e, y_{i1s}^e\right) = y_{n1s}^e + y_{i1s}^e = 0, \forall s = 1, 2, \tag{A.219}$$

where $y_{n00}^e \in \mathbb{R}_+$ ($y_{i00}^e \in \mathbb{R}_-$) and $y_{n1s}^e \in \mathbb{R}_+$ ($y_{i1s}^e \in \mathbb{R}_-$) are the outputs in unit of capital goods (inputs in unit of consumption goods) in period $t = 0$, and at state s in period $t = 1$, respectively. These specifications with profit maximization limit the price of capital not to be larger than 1 at any point in time. On the other hand, the capital investment is irreversible; that is, it is not feasible to directly turn a capital good into consumption goods. This irreversibility leads to fire sales, which can cause the price of capital to be significantly below one.

In addition, each entrepreneur has access to an entrepreneurial production technology, which transforms y^e_{n00} units of the capital goods in period $t = 0$ into $a_s y^e_{n00}$ units of the consumption goods in period $t = 1$, where $s = 1, 2$ is the aggregate state. This technology can be represented by the following production function:

$$F^e_{c1s}\left(y^e_{c1s}, y^e_{n00}\right) = y^e_{c1s} - a_s y^e_{n00} = 0, \forall s = 1, 2, \tag{A.220}$$

where y^e_{c1s} is the output in unit of consumption goods at state $s = 1, 2$ in period $t = 1$.

The capital must be repaired at the cost $\gamma > 0$ units of consumption goods at state $s = 1, 2$ in period $t = 1$ per unit of capital chosen to be repaired. Non-repaired part will be fully depreciated. This maintenance technology can be represented by the following production function:

$$F^e_{r1s}\left(y^e_{o1s}, y^e_{r1s}\right) = \gamma y^e_{o1s} + y^e_{r1s} = 0, \forall s = 1, 2, \tag{A.221}$$

where $y^e_{o1s} \in \mathbb{R}_+$ is the output in unit of (old) capital goods at state s in period $t = 1$ from this maintenance process, and $y^e_{r1s} \in \mathbb{R}_-$ is the input in unit of consumption goods for the maintenance process. The production technology also requires that old repaired capital cannot be larger (in absolute value) than the original capital from period $t = 0$, i.e.,

$$F^e_{p1s}\left(y^e_{o1s}, y^e_{r1s}\right) = y^e_{o1s} + y^e_{r1s} \geq 0, \forall s = 1, 2. \tag{A.222}$$

Further, an entrepreneur can use all capital available in period $t = 1$, y^e_{k1s}, to produce $A y^e_{k1s}$ units of consumption goods in period $t = 2$, with $A > 1$. This technology can be represented by the following production function:

$$F^e_{c2s}\left(y^e_{c2s}, y^e_{k1s}\right) = y^e_{c2s} - A y^e_{k1s} = 0, \forall s = 1, 2, \tag{A.223}$$

where y^e_{c2s} is the output in unit of consumption goods at state $s = 1, 2$ in period $t = 1$.

Each consumer owns a traditional production technology, which produces consumption goods in period $t = 2$ using capital goods in period $t = 1$, y^c_{k1s}, as the input. The traditional technology is represented by the production function $f\left(y^c_{k1s}\right)$, which is assumed to be increasing, strictly concave, twice differentiable, and satisfies the following properties $f(0) = 0, f'(0) = 1, f'\left(y^c_{1s}\right) \geq \bar{q}$. Strict concavity and $f'(0) = 1$ assumptions imply that consumers would not produced (new) capital using technology (A.218)-(A.219) even if they were be able to do so. They will own capital only when there is fire sales, under which the price of capital wold be below one. The capital good is fully depreciated at the end of the last period $t = 2$. This traditional technology can be represented by the following production function:

$$F^c_{c2s}\left(y^c_{c2s}, y^c_{k1s}\right) = y^c_{c2s} - f\left(y^c_{k1s}\right) = 0, \forall s = 1, 2, \tag{A.224}$$

where y^c_{c2s} is the output in unit of consumption goods in period $t = 2$.

To sum up, there are $O = 13$ production functions in this model.

G.1.6 Basic Ingredients for the Hidden Information with Retrading Economy

This is an economy with unobserved states or preference/liquidity shocks and retrading possibilities (e.g., Allen and Gale, 2004; Diamond and Dybvig, 1983; Farhi et al., 2009; Jacklin, 1987). Similar to the moral hazard problem, if there were no retrading possibility, then the Prescott-Townsend equilibria would have been equivalent to Pareto optima. However, this liquidity problem features externalities when agents can trade in spot/private markets ex-post creating the interaction of binding incentive constraints and the spot prices. As in Prescott and Townsend (1984b) and Farhi et al. (2009), we focus only on incentive compatible allocations (rather than sequential service constraints and no bank runs).

There is a continuum of ex-ante identical agents with total mass 1, i.e., no diversity in types so trivially $\alpha^1 = 1$. There are three periods, $t = 0, 1, 2$. There is one physical commodity in each period $t = 1, 2$. Each agent is endowed with e units of the good in the contracting period, $t = 0$, and this will be an input into production functions.

Let η be a ex-post preference/liquidity shock which defines a (idiosyncratic) state in this model. There is no loss of generality to assume that there are a finite Q states, $\eta \in Q$. The shock/state is drawn at $t = 1$ with $\pi(\eta)$ as the probability that an agent will receive η shock such that

$$\sum_{\eta=1}^{Q} \pi(\eta) = 1. \tag{A.225}$$

In this sense there is ex-post diversity. Henceforth, we represent an ex-post type of an agent by his shock η. The fraction of agents of ex-post type $\omega = \eta$ is $\zeta^1(\omega) = \pi(\eta)$. With a continuum of agents, we also interpret $\pi(\eta)$ as the fraction of agents receiving η shock.

To sum up, each state η has two dated commodities; that is, the physical good in period $t = 1$ or good 1, and the physical good in period $t = 2$ or good 2. In addition, an investment decision at $t = 0$, ρ, is also a commodity. Therefore, there are $L = 2Q + 1$ commodities in this model.

The utility function conditional on a shock η is given by $u(c_1, c_2, \eta)$, where (c_1, c_2) is the vector of consumption allocations in period $t = 1$ and $t = 2$, respectively. For example, in the Diamond-Dybvig model, the shock will dictate if an agent would like to consume now or later. The utility function is assumed to be differentiable, concave, increasing in c_1 and c_2, and satisfies the usual Inada conditions with respect to c_1 and c_2. The discounted expected utility of an agent is define by:

$$U(\mathbf{c}) = \sum_{\eta} \pi(\eta) u(c_1(\eta), c_2(\eta), \eta). \tag{A.226}$$

Following the literature, there are two technologies or assets. First, the short-term asset is a storage technology, whose return from t to $t+1$ is R_1, i.e., saving one unit of the good today at t will return R_1 units of the good in the next period at $t+1$, $t = 0, 1$. The second asset is the long-term asset. The long-term investment must be taken at $t = 0$, and its return R_2 will be realized at $t = 2$. We assume that the long-term asset is more productive than the short-term asset, i..e, $R_2 > R_1$. For simplicity, returns here are deterministic, i.e., no aggregate shocks.

There is no loss of generality to assume that an agent must decide how much to invest in the short-term and the long-term assets at the beginning $t = 0$. Let ρ be the fraction of initial endowment invested in the short-term asset; that is, ρe is the total amount invested in the short-term asset, and $(1-\rho)e$ is the total amount invested in the long-term asset. In addition, we assume that there is no option to liquidate at $t = 1$, and there is no short term investment between $t = 1$ and $t = 2$ without loss of generality[13]. The production functions of short-term asset, F_1 (between $t = 0$ and $t = 1$), and long-term asset F_2 (between $t = 0$ and $t = 2$) are as follows:

$$F_1(\rho e, y_1) = y_1 - R_1 \rho e = 0, \tag{A.227}$$

$$F_2((1-\rho)e, y_2) = y_2 - R_2(1-\rho)e = 0, \tag{A.228}$$

where y_t is an output in unit of the physical good in period $t = 1, 2$ regardless of the state of nature. Note however that outputs do not really vary with the preference/liquidity shocks since the shocks η are liquidity, not productivity shocks and the distribution of shocks in the population is constant. To sum up, there are $O = 2$ production functions.

[13] This economy is equivalent to the one in Diamond and Dybvig (1983) where banks invest in the long-term asset only, and then liquidate a fraction of the projects at $t = 1$. In Allen and Gale (2004) with stochastic returns, some short term investment may be necessary at $t = 1$.

G.2 Market Structure: Security and Spot Markets

There are J securities. Let $\theta_j^h \in \mathbb{R}$ denote the amount of security j acquired (negative if sold) by an agent of type h, and $\mathbf{D}_j = [D_{j\ell}]_{\ell=1}^L \in \mathbb{R}_+^L$ denote its payoff vector. Thus securities have payoffs of goods in the L-dimensional space of underlying commodities. Notationally, let $\mathbf{D} = [\mathbf{D}_j]_{j=1}^J$ be the payoff matrix of all securities. Let $\mathbf{Q} \in \mathbb{R}_+^J$ be the price vector of all securities, that is, $Q_j \geq 0$ for $j = 1, \ldots, J$.

In addition, agents can trade in each of M spot markets of subsets of commodities. Let $L^m \subset L$ be the subset of commodities that can be traded in spot markets m. Let $\boldsymbol{\tau}^h$ denote the set of trades in these markets with $\tau_{\ell m}^h$ denoting the amount of good ℓ in market L^m acquired (negative if surrendered) by an agent of type h. Note again that these spot trades $\tau_{\ell m}^h$ are restricted to be traded with commodities in L^m only. Let $\mathbf{p}_m \equiv [p_{\ell m}]_{\ell \in L^m} \in \mathbb{R}_+^{L^m}$ be the price vector of commodities in L^m.

The relationship between consumption, endowments, securities, spot trades, and outputs for an agent of type h is defined implicitly by

$$g^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) = 0. \tag{A.229}$$

These will be obvious identities or accounting formulas in the examples which follow. This condition is generally multidimensional vector with dimension N, i.e.,

$$g^h \equiv \left(g_n^h\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right)\right)_{n=1}^N.$$

G.2.1 Market Structure for the Collateral Economy

Let $\theta_{\ell s}^h$ denote securities paying in good $\ell = 1, 2$ in state s or net transfers of good $\ell = 1, 2$ in state s acquired by an agent type h. If this is negative, it is a promise to pay. Also $\theta_{\ell 0}^h$ is spot purchase of good ℓ at $t = 0$ but for convenience we refer to this as a security trade. Thus there are $J = 2(1 + S)$ securities. Let $Q_{\ell 0}$ and $Q_{\ell s}$ denote the security (spot) price of good ℓ at period $t = 0$ and the price of a security paying in good ℓ in state s, respectively. We take good $\ell = 1$ as the numeraire.

Let $\tau_{\ell s}^h$ denote spot trade amount of good $\ell = 1, 2$ in spot markets in state s, L^s ($m = s$ here), acquired by an agent of ex-ante type h. With abuse of notation, let $\tau_{\ell 0}^h$ denote spot trade amount of good ℓ in spot markets L^0 in period $t = 0$ acquired by an agent of ex-ante type h. Each spot market has two commodities, namely good 1 and good 2, i.e., $L^m = 2$ for all $m = 0, 1, \ldots, S$. There are $M = S+1$ spot markets here. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_s denote the spot-market-clearing price of good 2 in each spot market L^s.

The consumption-relationship constraints in this case are defined as follows:

$$g^h_{\ell s}\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h\right) = e^h_{\ell s} + y^h_{\ell s} + \theta^h_{\ell s} + \tau^h_{\ell s} - c^h_{\ell s} = 0, \text{ for } \ell = 1, 2; s = 0, 1, \ldots, S, \quad \text{(A.230)}$$

where we set $y^h_{10} = 0$ and $y^h_{1s} = 0$ to represent the fact that good 1 cannot be stored. There are $N = 2(1+S)$ consumption-relationship constraints. As proved in Kilenthong and Townsend (2014), with complete collateralized contracts, there is no need for restricted/spot trades $\boldsymbol{\tau}$ in this case. All trades can be accomplished in ex-ante security markets. As a result, the consumption-relationship constraints can be rewritten as follows:

$$g^h_{\ell s}\left(\mathbf{c}^h, \mathbf{e}^h, \boldsymbol{\theta}^h, \mathbf{y}^h\right) = e^h_{\ell s} + y^h_{\ell s} + \theta^h_{\ell s} - c^h_{\ell s} = 0, \text{ for } \ell = 1, 2; s = 0, 1, \ldots, S. \quad \text{(A.231)}$$

Nevertheless, we can define what the ex-post spot price p_s would be that would clear these markets (without active trade).

G.2.2 Market Structure for the Exogenous Incomplete Markets Economy

There are $J < S$ securities available for purchase or sell in the first period $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets where D_{js} be the payoff of asset j in unit of good 1 (the numeraire good) in state s in the second period $t = 1$, $s = 1, 2, \ldots, S$. Let θ^h_j denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$, and Q_j denote the price of security j. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is again, $J < S$. This is crucial.

Let $\tau^h_{\ell s}$ denote spot trade amount of good $\ell = 1, 2$ in spot markets in state s, L^s ($m = s$ here), acquired by an agent of ex-ante type h. With abuse of notation, let $\tau^h_{\ell 0}$ denote spot trade amount of good $\ell = 1, 2$ in spot markets L^0 in period $t = 0$ acquired by an agent of ex-ante type h. Each spot market has two commodities, namely good 1 and good 2, i.e., $L^m = 2$ for all $m = 0, 1, \ldots, S$. There are $M = S + 1$ spot markets here. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_0 and p_s denote the spot-market-clearing price of good 2 in spot market L^0 in period $t = 0$, and the spot-market-clearing price of good 2 in spot market L^s at state s in period $t = 1$, respectively.

The consumption-relationship functions in the first period $t = 0$ is defined as follows:

$$g_{\ell}^{h}\left(\mathbf{c}^{h}, \mathbf{e}^{h}, \boldsymbol{\theta}^{h}\right) = e_{\ell 0}^{h} + \tau_{\ell 0}^{h} - c_{\ell 0}^{h} = 0, \text{ for } \ell = 1, 2. \tag{A.232}$$

The consumption-relationship function for good 1 and good 2, respectively, in the state s in the second period is defined as follows:

$$g_{1+2s}^{h}\left(\mathbf{c}^{h}, \mathbf{e}^{h}, \boldsymbol{\theta}^{h}, \boldsymbol{\tau}^{h}\right) = e_{1s}^{h} + \sum_{j} D_{js}\theta_{j}^{h} + \tau_{1s}^{h} - c_{1s}^{h} = 0, \text{ for } s = 1, \ldots, S, \tag{A.233}$$

$$g_{2+2s}^{h}\left(\mathbf{c}^{h}, \mathbf{e}^{h}, \boldsymbol{\theta}^{h}, \boldsymbol{\tau}^{h}\right) = e_{2s}^{h} + \tau_{2s}^{h} - c_{2s}^{h} = 0, \text{ for } s = 1, \ldots, S. \tag{A.234}$$

Note here there will be active spot market trades $\boldsymbol{\tau}$ to support the equilibrium allocation. To sum up, there are $N = 2(1 + S)$ consumption-relationship constraints.

G.2.3 Market Structure the Moral Hazard with Retrading Economy

To be consistent with the general model, one can imagine that there are state-contingent securities paying in good $\ell = 1, 2$ when state/output is $\mathbf{q}$ and the recommended action is a, namely $\theta_{\ell}(\mathbf{q}, a)$. That is, security j is indexed by $\mathbf{q}$, ℓ, and a. Even though there are $J = 2QA$ securities available to trade, each agent can trade only $2Q$ securities depending on his recommended action a only. In particular, an agent recommended action a will be able to trade only securities $\theta_a \equiv [\theta_{\ell}(\mathbf{q}, a)]_{\ell,\mathbf{q}}$. Let $Q_{\ell}(\mathbf{q}, a)$ denote the price of a security paying in good ℓ conditional on output $\mathbf{q}$ and recommended action a. Recall that actual action $a' = a$ is an input of the production technology, and there is no loss of generality to assume that it is non-tradable. Therefore, there is no price for that commodity action. Note also that as in the literature, these equilibrium securities prices are fair prices.

There is the possibility of retrade in ex post spot markets. One can think of two subperiods: the first with the application of inputs, securities, and production; the second for output and possible retrading with final consumption. Without aggregate uncertainty, there is only one set of spot markets ($M = 1$) for good 1 and good 2 ($L^1 = 2$). Let $\tau_{\ell}(\mathbf{q}, a)$ be spot trade of an agent of ex-post type $(\mathbf{q}, a)$ when a is both the recommended and taken action. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p denote the spot-market-clearing price of good 2, which depends on agents' action a (recommended and taken) and securities $\theta_a = [\theta_{\ell}(\mathbf{q}, a)]_{\ell,\mathbf{q}}$ as a function of recommended action a (as if the markets can be partitioned by action a); that is, $p = p(\theta_a, a)$.

As in Kilenthong and Townsend (2011) and the collateral example in Section G.2.1, the spot markets are redundant (with complete contracts), however. Anything that can be done with spot markets can be done without them with altered security holdings. Therefore, we can omit spot trades, henceforth, though there is still an implicit shadow spot price. The consumption-relationship in this case is defined as follows:

$$g_{\ell \mathbf{q} a} = q_\ell + \theta_\ell(\mathbf{q}, a) - c_\ell(\mathbf{q}, a) = 0, \forall \mathbf{q}, a; \ell = 1, 2. \tag{A.235}$$

There are $N = 2QA$ consumption-relationship constraints.

G.2.4 Market Structure the Liquidity Constrained Economy

To be consistent with the general model, there is no security in this model; that is, $J = 0$. All trades occur in the spot markets. There are 2 sets of spot markets in period $t = 1$ and $t = 2$; that is, $M = 2$. Agents can trade storage claim τ_{f1}^h and building services τ_b^h in the spot markets in period $t = 1$ at price p_b; that is, there are two commodities in the spot markets in period $t = 1$ ($L^1 = 2$). Similarly, agents can trade storage claim τ_{f2}^h and doctor services τ_d^h in the spot markets in period $t = 2$ at price p_d; that is, there are two commodities in the spot markets in period $t = 2$ ($L^2 = 2$).

The consumption-relationship constraints are as follows:

$$g_b^h \left(b^h, y_b^h, \tau_b^h \right) = b^h - \left(y_b^h + \tau_b^h \right) = 0, \forall h, \tag{A.236}$$

$$g_d^h \left(d^h, y_d^h, \tau_d^h \right) = d^h - \left(y_d^h + \tau_d^h \right) = 0, \forall h, \tag{A.237}$$

$$g_w^h \left(w^h, y_{i3}^h, y_{s3}^h \right) = w_3^h - y_{i3}^h - y_{s3}^h = 0, \forall h, \tag{A.238}$$

$$g_{ft}^h \left(f_t^h, f_{t-1}^h, \tau_{ft}^h \right) = f_t^h - f_{t-1}^h - \tau_{ft}^h = 0, \forall h; t = 1, 2. \tag{A.239}$$

To sum up, there are $N = 5$ consumption-relationship constraints.

G.2.5 Market Structure the Fire Sales Economy

Let θ_0^h and θ_{ts}^h denote securities paying in unit of consumption goods in period $t = 0$, and securities paying in consumption goods at state $s = 1, 2$ in period $t = 1, 2$ acquired by an agent type $h = c, e$, respectively. There are $J = 5$ securities. Let Q_{ts} denote the contract/security price of a security paying in consumption goods at state $s = 1, 2$ in period $t = 1, 2$, and $Q_0 = 1$ the price of the contract/security paying in unit of consumption goods in period $t = 0$, which is the numeraire good.

There are also spot markets at each state $s = 1, 2$ in period $t = 1$. There are $L^m = 2$ commodities in each spot markets $m = 1, 2$. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p_s denote the spot-market-clearing price of good 2. As in Lorenzoni (2008), the market clearing conditions for the spot markets in each state s (for τ^h_{ks} and τ^h_{cs}) imply that the spot price p_s is determined by capital input for the traditional technology y^c_{k1s}, i.e., $p_s = f'\left(y^c_{k1s}\right)$.

The consumption-relationship functions for consumers and entrepreneurs are given by

$$g^c_{c0}\left(e, \theta^c_0, c^c_0\right) = e + \theta^c_0 - c^c_0 = 0, \tag{A.240}$$

$$g^c_{c1s}\left(e, \theta^c_{1s}, \tau^c_{cs}, c^c_{1s}\right) = e + \theta^c_{1s} + \tau^c_{cs} - c^c_{1s} = 0, \forall s = 1, 2, \tag{A.241}$$

$$g^c_{c2s}\left(e, \theta^c_{2s}, y^c_{c2s}, c^c_{2s}\right) = e + \theta^c_{2s} + y^c_{c2s} - c^c_{2s} = 0, \forall s = 1, 2, \tag{A.242}$$

$$g^c_{ks}\left(\tau^c_{ks}, y^c_{k1s}\right) = \tau^c_{ks} - y^c_{k1s} = 0, \forall s = 1, 2, \tag{A.243}$$

$$g^e_{c0}\left(n, \theta^e_0, y^e_{i00}\right) = n + \theta^e_0 - y^e_{i00} = 0, \tag{A.244}$$

$$g^e_{c1s}\left(\theta^e_{1s}, \tau^e_{cs}, y^e_{c1s}, y^e_{r1s}, y^e_{i1s}\right) = \theta^e_{1s} + \tau^e_{cs} + y^e_{c1s} + y^e_{r1s} + y^e_{i1s} = 0, \forall s = 1, 2, \tag{A.245}$$

$$g^e_{ks}\left(y^e_{n1s}, y^e_{o1s}, \tau^e_{ks}, y^e_{k1s}\right) = y^e_{n1s} + y^e_{o1s} + \tau^e_{ks} - y^e_{k1s} = 0, \forall s = 1, 2, \tag{A.246}$$

$$g^e_{c2s}\left(y^e_{c2s}, \theta^e_{2s}, c^e_{2s}\right) = y^e_{c2s} + \theta^e_{2s} - c^e_{2s} = 0, \forall s = 1, 2, \tag{A.247}$$

where a function with superscript "c" ("e") is a consumption-relationship function for consumers (for entrepreneurs). There are $N = 14$ consumption-relationship conditions.

G.2.6 Market Structure the Hidden Information with Retrading Economy

To be consistent with the general model, one can imagine that there are state-contingent securities $\theta_t(\eta')$ paying the single good at $t = 1, 2$ conditional on reported shock/state $\eta' \in Q$. That is, security j is indexed by t and η'. There are $J = 2Q$ (the number of states times the number of dates) securities available.

There is the possibility of retrade in ex post spot markets as in the moral hazard with retrading economy above. Without aggregate uncertainty, there is only one set of spot markets ($M = 1$) for good 1 and good 2 ($L^m = 2$), in which everyone participates. Let $\tau_t(\eta)$ be spot trade of an agent of ex-post type η when η is both truthfully reported and realized shock. We set the spot-market-clearing price of good 1 equal to one (the numeraire good), and let p denote the spot-market-clearing price of good 2, which depends on securities $\boldsymbol{\theta} = [\theta_t(\eta)]_{t,\eta}$ and investment decision ρ; that is, $p = p(\theta, \rho)$.

As in the moral hazard with retrading economy, the spot markets are redundant (with complete contracts), however. Anything that can be done with spot markets can be done without them with altered security holdings. Therefore, we can omit spot trades, henceforth, though there is still an implicit shadow spot price. The consumption-relationship in this case is defined as follows:

$$g_{t\eta'} = y_t + \theta_t(\eta') - c_t(\eta') = 0, \forall \eta'; t = 1, 2. \tag{A.248}$$

There are $N = 2Q$ consumption-relationship constraints.

G.3 Trade Frictions: Obstacle-to-Trade Constraints

There are I sets of obstacle-to-trade constraints, each of which contains potentially multiple conditions, indexed by $a = 1, \ldots, A$ and $a' = 1, \ldots, A$. Each set of obstacle-to-trade constraints $i = 1, 2, ...I$ depends on the spot prices of a particular subset of commodities $\mathbf{p}^i$ or the prices of a particular subset of securities denoted $\mathbf{Q}^i$, or both. Each can depend on the same set of prices $\left(\mathbf{p}^i, \mathbf{Q}^i\right)$. In their general form, each obstacle to trade constraint (a, a') in set i can be written as:

$$C^h_{i,a,a'}\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h, \mathbf{p}^i, \mathbf{Q}^i\right) \geq 0, \text{ for } i = 1, \ldots, I; a \in A; a' \in A. \tag{A.249}$$

These obstacle-to-trade constraints could be in the form of collateral constraints, retrading in exogenous incomplete-market constraints, incentive compatibility constraints under moral hazard with retrading, incentive compatibility constraints under hidden information with retrading, liquidity constraints, and no-default constraints. The total number of obstacle-to-trade constraints is $V \leq IA^2$, where the inequality results from the fact that the maximum number of constraints for each agent type is IA^2 but it is possible for some types to have less. Note that an action as in a moral hazard model, or privately observed state indexes the commodities, and therefore is included in $\mathbf{c}^h$. In addition to actions or preference shocks, the index (a, a') also denote each individual constraint in each set i of the obstacle-to-trade constraints sharing the same set of prices $\left(\mathbf{p}^i, \mathbf{Q}^i\right)$.

The dependency on market-clearing prices of these obstacle-to-trade constraints is the source of price externalities in this paper. Most of the literature focuses only on the dependency on the restricted/spot prices. This paper explicitly puts security prices into the constraints in order to emphasize that price externalities could arise even when we shut down the spot markets. In other words, the spot markets/prices are not fundamental to the externality problem. It is an obstacle to trade itself, which can not be removed, that is key to the problem. As shown in the collateral economy below, one can get rid of the spot markets there since they are redundant. The collateral constraints (the need to back promises by collateral) then depend on security prices only, but the price externality still occurs.

G.3.1 Trade Frictions for the Collateral Economy

As in Kilenthong and Townsend (2014), the collateral constraints or obstacle-to-trade constraints state that the value of collateral y^h_{2s} must weakly exceed value of promises to pay $\left(\theta^h_{1s}, \theta^h_{2s}\right)$:

$$p_s y^h_{2s} \geq p_s \left(-\theta^h_{2s}\right) + \left(-\theta^h_{1s}\right), \text{for } s = 1, \ldots, S, \tag{A.250}$$

which can be rewritten as follow:

$$p_s \left(y^h_{2s} + \theta^h_{2s}\right) + \theta^h_{1s} \quad \geq \quad 0, \text{ for } s = 1, \ldots, S, \tag{A.251}$$

where again p_s is the spot price of good 2 in units of good 1 in state s.

But as mentioned earlier, these collateral constraints can be rewritten in terms of security prices as following:

$$C^h_s \left(\theta^h, \mathbf{y}^h, \mathbf{Q}_s\right) = Q_{2s} \left(y^h_{2s} + \theta^h_{2s}\right) + Q_{1s}\theta^h_{1s} \quad \geq \quad 0, \text{ for } s = 1, \ldots, S, \tag{A.252}$$

which results from the fact that, with complete state contingent contracts at $t = 0$ and the possibility of retrading, the spot price ratio p_s equals to the ratio of security prices $\frac{Q_{2s}}{Q_{1s}}$. This formulation emphasizes that we can shut down the spot markets, but the collateral constraints still depend on security prices, which still generate externalities. In other words, the spot markets/prices are not fundamental to the externality problem. It is an obstacle to trade itself, which can not be removed, that is key to the problem.

Each agent of type h faces $I = S$ sets of obstacle-to-trade constraints, each of which contains only one constraint, i.e., technically $A = 1$. Therefore, there are $V = S$ obstacle-to-trade constraints in total.

G.3.2 Trade Frictions for the Exogenous Incomplete Markets Economy

The obstacle-to-trade or spot-budget constraint for an agent of type h in each state s is simply the budget constraint in that state:

$$C^h_s \left(\boldsymbol{\tau}^h_s, \mathbf{p}\right) = \tau^h_{1s} + p_s \tau^h_{2s} = 0, \text{ for } s = 1, \ldots, S, \tag{A.253}$$

Note that the spot price p_s is determined by pre-trade position of endowments and securities where endowments are exogenous but securities are endogenous.

Each agent of type h faces $I = 1$ set of obstacle-to-trade constraints, which contains $A = S$ constraints. Therefore, there are $V = S$ obstacle-to-trade constraints in total.

G.3.3 Trade Frictions for the Moral Hazard with Retrading Economy

The possibility of retrade in ex post spot markets creates obstacle to trade in this model. With the possibility of retrade, the ex-post utility maximization problem of an agent who was recommended action a receiving compensation $(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a))$, but took action a' when the spot market price is p is as follows:

$$v\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a', p\right) = \max_{\tau_1, \tau_2} u\left(c_1(\mathbf{q}, a) + \tau_1, c_2(\mathbf{q}, a) + \tau_2, a'\right) \tag{A.254}$$

subject to the budget constraint:

$$\tau_1 + p\tau_2 = 0, \tag{A.255}$$

taking spot-market-clearing price p as given.

As in Kilenthong and Townsend (2011), the possibility of retrade in ex post spot markets and the moral hazard problem imply that the incentive compatibility constraints (IC) are as following: $\forall a, a'$,

$$C_{1,a,a'}\left(\mathbf{c}, p\right) = \sum_{\mathbf{q}} u\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a\right) f(\mathbf{q}|a) - \sum_{\mathbf{q}} v\left(c_1(\mathbf{q}, a), c_2(\mathbf{q}, a), a', p\right) f(\mathbf{q}|a') \geq 0, \tag{A.256}$$

Here the agent takes the recommended action a and so $a = a'$. There is only one set of obstacle-to-trade constraints, $I = 1$, and there are A^2 constraints for this one i. Therefore, there are $V = A^2$ incentive compatibility constraints in total.

G.3.4 Trade Frictions for the Liquidity Constrained Economy

The obstacle-to-trade or spot market constraints for an agent type $h = b, d$ in period t are as follows:

$$C_1^h\left(\tau_{f1}^h, \tau_b^h, \mathbf{p}\right) = \tau_{f1}^h + p_b\tau_b^h = 0, \forall h = b, d, \tag{A.257}$$

$$C_2^h\left(\tau_{f2}^h, \tau_d^h, \mathbf{p}\right) = \tau_{f2}^h + p_d\tau_d^h = 0, \forall h = b, d, \tag{A.258}$$

where p_b and p_d are the spot-market-clearing prices of building and doctor services in period $t = 1$ and $t = 2$, respectively; that is, p_b is such that $\sum_{h=b,d} \alpha^h \tau_b^h = 0$, and vice versa. Note that the spot price p_b and p_d are determined by storage positions of all agents which are endogenous. Each agent of type h faces $I = 1$ set of obstacle-to-trade constraints, which contains $A = 2$ constraints. Therefore, there are $V = 2$ obstacle-to-trade constraints in total.

G.3.5 Trade Frictions for the Fire Sales Economy

Each consumer faces the following sets of obstacle-to-trade constraints. First, the participation constraint for a consumer is given by

$$C^c_{pc}\left(\theta^c_0, \theta^c_{1s}, \theta^c_{2s}\right) \quad = \quad \theta^c_0 + \sum_s \pi_s \left(\theta^c_{1s} + \theta^c_{2s}\right) \geq 0. \qquad \text{(A.259)}$$

This constraint states that a consumer would not enter the contract at period $t = 0$ and would be at autarky unless the contract offers a non-negative expected return from $t = 0$ to $t = 2$. Second, the no-default conditions for a consumer are as follows:

$$C^c_{cd1}\left(\theta^c_{1s}, \theta^c_{2s}\right) \quad = \quad \theta^c_{1s} + \theta^c_{2s} \geq 0, \forall s = 1, 2, \qquad \text{(A.260)}$$

$$C^c_{cd2}\left(\theta^c_{2s}\right) \quad = \quad \theta^c_{2s} \geq 0, \forall s = 1, 2. \qquad \text{(A.261)}$$

These constraints imply that a consumer would default (not pay when $\theta^c_{ts} < 0$) at state $s = 1, 2$ in period $t = 1, 2$ unless the return from that period on is non-negative.

Each entrepreneur faces the following obstacle-to-trade constraints or no-default conditions:

$$C^e_{ed1}\left(y^e_{n00}, \theta^e_{1s}, \theta^e_{2s}, p_s\right) = \left(\eta a_s + \max\left\{p_s - \gamma, 0\right\}\right) y^e_{n00} + \theta^e_{1s} + \theta^e_{2s} \geq 0, \forall s = 1, 2, \qquad \text{(A.262)}$$

$$C^e_{ed2}\left(y^e_{k1s}, \theta^e_{2s}\right) = \eta A y^e_{k1s} + \theta^e_{2s} \geq 0, \forall s = 1, 2, \qquad \text{(A.263)}$$

where $1 - \eta \in (0, 1)$ is the fraction of the firm's current profit that the entrepreneur could keep if he decided to default. Constraints (A.262) imply that the entrepreneur is better off not defaulting at state $s = 1, 2$ in period $t = 1$. In particular, he would get $(1 - \eta)\, a_s y^e_{n00}$ if he defaulted. On the other hand, his net income would be $\left(a_s + \max\left\{p_s - \gamma, 0\right\}\right) y^e_{n00} + \theta^e_{1s} + \theta^e_{2s}$ in case of no default. Similarly, constraints (A.263) imply that net income of the entrepreneur at state $s = 1, 2$ in period $t = 2$ in case of no default, $A y^e_{k1s} + \theta^e_{2s}$, is larger than his net income in case of default, $(1 - \eta)\, A y^e_{k1s}$.

In addition, both agent types also face the following spot market budget constraints:

$$C^h_{spot}\left(\tau^h_{cs}, \tau^h_{ks}, p_s\right) \quad = \quad \tau^h_{cs} + p_s \tau^h_{ks} = 0, \forall h = c, e; s = 1, 2, \qquad \text{(A.264)}$$

To be consistent with the general model, there are $I = 7$ sets of obstacle-to-trade constraints for the consumer c and 6 sets of obstacle-to-trade constraints for the entrepreneur e. Each set contains only one constraint, i.e., $A = 1$. Therefore, there are 7 obstacle-to-trade constraints in total for the consumer c and 6 obstacle-to-trade constraints in total for the entrepreneur e.

It is worthy of emphasis that the spot market budget constraints (A.264) are not the sources of the externality here because this model has a complete contingent contracting structure. See a similar result in Proposition A.1. On the other hand, the key obstacle-to-trade constraints that cause an inefficiency in this model is the first set of no-default conditions for an entrepreneur (A.262), which again depends on equilibrium prices which in turn are determined by collective ex-ante choices of the agents.

G.3.6 Trade Frictions for the Hidden Information with Retrading Economy

With abuse of notation, we refer to $(c_1(\eta'), c_2(\eta'))$ as pre-trade compensation condition on reported state/shock. Thus, the ex-post utility maximization problem at $t = 1$ of an agent who reported state η', realized state η, and received compensation $(c_1(\eta'), c_2(\eta'))$ is as follows:

$$v\left(c_1\left(\eta'\right), c_2\left(\eta'\right), \eta, p\right) = \max_{\tau_1, \tau_2} u\left(c_1\left(\eta'\right) + \tau_1, c_2\left(\eta'\right) + \tau_2, \eta\right) \tag{A.265}$$

subject to the budget constraint:

$$\tau_1 + p\tau_2 = 0, \tag{A.266}$$

taking spot price (interest rate) p as given.

In addition, the possibility of retrade in ex post spot markets and the hidden information problem imply that an incentive compatibility (IC) or obstacle-to-trade constraint:

$$C_{1,\eta,\eta'}\left(\mathbf{c}, p\right) = u\left(c_1\left(\eta\right), c_2\left(\eta\right), \eta\right) - v\left(c_1\left(\eta'\right), c_2\left(\eta'\right), \eta, p\right) \quad \geq \quad 0, \forall \eta, \eta'. \tag{A.267}$$

There are $I = Q^2$ constraints for each i, and therefore with only one i, there are $V = Q^2$ obstacle-to-trade constraints in total. This will be imposed so actual and reported states will be the same.

G.4 Competitive Equilibrium with Externalities of Prototypical Economies

This section presents the standard definitions of competitive equilibria that have externalities for a liquidity constrained economy (Hart and Zingales, 2013) and a fire sales economy (Lorenzoni, 2008). The definitions for an incomplete markets economy and a collateral economy are already displayed in online Appendix E.1 and F.1, respectively. The definitions of a moral hazard with retrading economy and a hidden information with retrading economy can be found in Kilenthong and Townsend (2011).

G.4.1 Competitive Equilibrium for the Liquidity-Constrained Economy

Definition A.9. *A competitive equilibrium with liquidity constraints is a specification of allocation* $\left(f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, b^h, d^h, w_3^h, \mathbf{y}^h\right)_{h=b,d}$ *and prices* (p_b, p_d) *such that*

(i) *for each h,* $\left(f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, \mathbf{y}^h\right)$ *solves the utility maximization problem*

$$\max_{f_0^h, l^h, \tau_b^h, \tau_d^h, \tau_{f1}^h, \tau_{f2}^h, \mathbf{y}^h} w_3^h + \delta_b^h b^h + \left(1 - \delta_d^h\right) d^h - \frac{\left(l^h\right)^2}{2} \tag{A.268}$$

subject to the production constraints (A.212)-(A.215), the consumption-relationship constraints (A.236)-(A.239), and the obstacle-to-trade constraints (A.257)-(A.258), taking prices (p_b, p_d) *as given;*

(ii) *markets clear for storage claims in period* $t = 1$

$$\sum_h \alpha^h \tau_{f1}^h = 0, \tag{A.269}$$

markets clear for building services in period $t = 1$

$$\sum_h \alpha^h \tau_b^h = 0, \tag{A.270}$$

markets clear for storage claims in period $t = 2$

$$\sum_h \alpha^h \tau_{f2}^h = 0, \tag{A.271}$$

markets clear for doctor services in period $t = 2$

$$\sum_h \alpha^h \tau_d^h = 0, \tag{A.272}$$

G.4.2 Competitive Equilibrium for the Fire Sales Economy

Definition A.10. *A competitive equilibrium with fire sales is a specification of allocation* $\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h\right)_{h=c,e}$ *and prices* $(p_s, Q_{ts})_s$ *such that*

(i) *for each consumer c,* $(\mathbf{c}^c, \theta^c, \tau^c, \mathbf{y}^c)$ *solves the utility maximization problem*

$$\max_{\mathbf{c}^c, \theta^c, \tau^c, \mathbf{y}^c} c_0^c + \sum_{s=1,2} \pi_s \left(c_{1s}^c + c_{2s}^c\right) \tag{A.273}$$

subject to the budget constraints

$$\theta_0^c + \sum_{s,t} Q_{ts} \theta_{ts}^c \quad \leq \quad 0, \tag{A.274}$$

the production constraints (A.224), the consumption-relationship constraints (A.240)-(A.243), and the obstacle-to-trade constraints (A.259)-(A.261), (A.264), taking prices (p_s, Q_{ts}) *as given;*

(ii) for each entrepreneur e, $(\mathbf{c}^e, \theta^e, \tau^e, \mathbf{y}^e)$ *solves the utility maximization problem*

$$\max_{\mathbf{c}^e, \theta^e, \tau^e, \mathbf{y}^e} \sum_{s=1,2} \pi_s c^e_{2s} \tag{A.275}$$

subject to the budget constraints

$$\theta^e_0 + \sum_{s,t} Q_{ts}\theta^e_{ts} \quad \leq \quad 0, \tag{A.276}$$

the production constraints (A.218)-(A.223), the consumption-relationship constraints (A.244)-(A.247), and the obstacle-to-trade constraints (A.262)-(A.263), (A.264), taking prices (p_s, Q_{ts}) *as given;*

(iii) markets clear in contract paying in period $t = 0$

$$\theta^c_0 + \theta^e_0 = 0, \tag{A.277}$$

markets clear in security paying at state s in period t

$$\theta^c_{ts} + \theta^e_{ts} = 0, \forall t = 1, 2; s = 1, 2, \tag{A.278}$$

markets clear in consumption good at state s in period $t = 1$

$$\tau^c_{cs} + \tau^e_{cs} = 0, \forall s = 1, 2, \tag{A.279}$$

markets clear in capital good at state s in period $t = 1$

$$\tau^c_{ks} + \tau^e_{ks} = 0, \forall s = 1, 2. \tag{A.280}$$

References

Acemoglu, D. and Simsek, A. (2012). Moral Hazard and Efficiency in General Equilibrium with Anonymous Trading. *Working Paper.*

Allen, F. and Gale, D. (2004). Financial Intermediaries and Markets. *Econometrica*, pages 1023–1061.

Diamond, D. and Dybvig, P. (1983). Bank Runs, Deposit Insurance, and Liquidity. *Journal of Political Economy*, 91(3):401.

Farhi, E., Golosov, M., and Tsyvinski, A. (2009). A Theory of Liquidity and Regulation of Financial Intermediation. *Review of Economic Studies*, 76(3):973–992.

Geanakoplos, J. and Polemarchakis, H. (1986). Existence, Regularity and Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete. *Uncertainty, Information and Communication: Essays in Honor of KJ Arrow*, 3:65–96.

Greenwald, B. and Stiglitz, J. (1986). Externalities in Economies with Imperfect Information and Incomplete Markets. *Quarterly Journal of Economics*, 101(2):229–264.

Hart, O. D. and Zingales, L. (2013). Inefficient Provision of Liquidity. Technical report, Working Paper.

Jacklin, C. (1987). Demand Deposits, Trading Restrictions, and Risk Sharing. *Contractual Arrangements for Intertemporal Trade*, pages 26–47.

Kilenthong, W. T. and Townsend, R. M. (2011). Information-Constrained Optima with Retrading: An Externality and its Market-Based Solution. *Journal of Economic Theory*, 146(3):1042–1077.

Kilenthong, W. T. and Townsend, R. M. (2014). Segregated Security Exchanges with Ex Ante Rights to Trade. Technical report, National Bureau of Economic Research.

Lorenzoni, G. (2008). Inefficient Credit Booms. *Review of Economic Studies*, 75(3):809–833.

Parthasarathy, K. R. (1967). *Probability measures on metric spaces*, volume 352. American Mathematical Soc.

Prescott, E. C. and Townsend, R. M. (1984a). General Competitive Analysis in an Economy with Private Information. *International Economic Review*, 25(1):1–20.

Prescott, E. C. and Townsend, R. M. (1984b). Pareto Optima and Competitive Equilibria with Adverse Selection and Moral Hazard. *Econometrica*, 52(1):21–45.

Prescott, E. S. and Townsend, R. M. (2005). Firms as Clubs in Walrasian Markets with Private Information: Technical Appendix. *mimeo Federal Reserve Bank of Richmond.*